Portfolio Rho-presentativity

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Abstract

Given an investment universe, we consider the vector $\rho(w)$ of correlations of all assets to a portfolio with weights w. This vector offers a representation equivalent to w and leads to the notion of ρ-presentative portfolio, that has a positive correlation, or exposure, to all assets. This class encompasses well-known portfolios, and comes as a complement to the notion of representative portfolio, that has positive amounts invested in all assets (e.g. the market-cap index).

We then introduce the concept of maximally ρ-presentative portfolios, that maximize under no particular constraint an aggregate exposure $f(\rho(w))$ to all assets, as measured by a symmetric, increasing and concave real-valued function f. We provide a basic characterization of these portfolios that is independent of f, show that they are long-only, form a union of polytopes and are quite rare. However, these portfolios offer a unifying framework for many well-known and possibly constrained long-only portfolios.

We also establish a correspondence between some classic long-only portfolio optimization problems constrained to have maximum weights and unconstrained problems whose objective involves $\rho(w)$. This extends the analytical results obtained in Jagannathan and Ma (2003) by characterizing explicitly the impact on the objective of these constraints often used by practitioners. Finally, we propose several theoretical and numerical applications that illustrate our results.

Keywords: Portfolio Construction, Correlation Optimization, Portfolio Constraints, Representative Portfolios, Portfolio Diversification, Rho-presentative, Maximally Rho-presentative.

1 Introduction

For more than a decade, new systematic and quantitative investment processes have attracted significant interest in the field of asset management. Without being exhaustive, we first briefly discuss how some of these strategies could be rediscovered in the context of the present article.

A simple portfolio delivering an exposure to the overall market that is different from the market capitalization weighted index is the Equally Weighted (EW) portfolio. This particular choice is not new, with [8] claiming that it dates back to 400 AD. Using volatility-adjusted weights as an *alternative representation* for a portfolio naturally leads to the concept of Equal-Volatility-Weighted portfolio (EVW). The relative contribution of each asset to the risk of a portfolio gives another way of representing it, and leads to the concept of portfolio that Equalizes these Risk Contributions or ERC (see [12, 17]). Following a different path, in Fundamental indexation [1], Arnott et al. proposed equity portfolios with weights proportional to key accounting measures such as sales, revenues and income. Such a portfolio is *representative* of a universe in the sense that it invests in each company in proportion to its “economic footprint” rather than its capitalization.

As we have seen, approaches such as EW, EVW, ERC and Fundamental Indexation emerge as a result of alternative representations for portfolio weights. As we shall see in this paper, (possibly constrained) optimized portfolios such as the Minimum Variance (MV, see [13]) and the Most Diversified Portfolio (MDP, see [6, 7]) can also be obtained through the representation of a portfolio by its correlations - or exposures - to all assets.

In practice, these long-only investment processes may be modified in a number of ways when reaching the implementation phase. An important consideration for portfolios that optimize a given objective function is for example the addition of maximum weight constraints. These are imposed by some regulators and implemented by practitioners, and it is important to understand their impact on the initial objective. In [9], it is shown that imposing such constraints for the MV problem is equivalent to minimizing an unconstrained variance objective using a modified covariance matrix. However, a limitation of the method is that the modified matrix depends on Lagrange multipliers that are either known after the MV optimization or determined through a numerically demanding optimization (a constrained max likelihood on matrices).

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1.1 Contributions of this Paper

A New Portfolio Representation via Correlations. The usual representation of a portfolio, that consists in reporting its weights in each asset of the investment universe, may not directly indicate to which degree the portfolio is exposed to a particular asset. For instance, not holding any financial stock does not necessarily mean no exposure to the financial sector. This observation prompts a new representation of portfolios: given a long-short portfolio w, we consider in Section 2 its *correlation spectrum* $\rho(w)$ that stores its correlations to the assets of the universe and prove that it carries all the information needed to recover w up to its leverage.

Notions of Representativity and ρ-presentativity. The capitalization weighted portfolio is usually viewed as being representative of the assets of its universe. Such *representative portfolios* have positive amounts invested across all assets, leading us to introduce in Section 3 the notion of *ρ-presentative portfolio*, that admits a positive correlation, or exposure, to all assets. Note that this definition is not limited to long-only portfolios.

Optimized long-only portfolios such as the MDP and the MV are ρ-presentative without necessarily holding all the assets. In contrast, the market capitalization portfolio, the EW or the EVW, invested across all assets, are not necessarily ρ-presentative. Both categories intersect as, for instance, the ERC resides in both.

Furthermore, portfolios that are ρ-presentative satisfy a fundamental property that is not true in general: the (not necessarily unique) least correlated long-only portfolio to a ρ-presentative portfolio is an asset. Using this key result, we prove that a long-only portfolio is always positively correlated to at least one asset and give a uniform lower bound for this correlation. We tackle a converse of this result in the next section.

Maximally ρ-presentative Portfolios. To complement the notion of ρ-presentative portfolio, we introduce in Section 4 the concept of *maximally ρ-presentative portfolio.* By definition, such a portfolio maximizes an aggregate exposure $f(\rho(w)) \in \mathbb{R}$ to all assets as measured by some increasing, symmetric and concave function f. We show that maximally ρ-presentative portfolios are long-only. To establish this result, the key is to prove that for any portfolio that is not long-only there always exists a long-only portfolio that is more correlated to all assets. In addition, we characterize explicitly this set of portfolios: these are essentially the long-only portfolios whose exposures form a non-increasing function of their volatility weighted weights. Furthermore, we show that these portfolios form a union of polytopes and that they are quite rare as, essentially, any permutation of a maximally ρ-presentative portfolio that is different from it is not maximally ρ-presentative.

However this new class encompasses many well-known portfolios. For instance the EVW is, amongst all long-short portfolios, the portfolio that maximizes its average correlation to all the assets. We also prove that the MDP is the portfolio that maximizes its minimal correlation to all the assets amongst all long-short portfolios. We refine this result by showing that the MDP maximizes its minimal correlation to all long-only factors, defined as factors that are replicable by possibly leveraged long-only portfolios of assets belonging to the universe. Similar results are established for the ERC, MV and EW portfolios.

On the Impact of Maximum Weight Constraints. We prove in Section 4.3 that a *constrained* MV or MDP problem with maximum weight $\frac{1}{k}$ is essentially equivalent to an *unconstrained* maximization of an average of the k smallest entries of $\rho(w)$. In addition to proving, for instance, that the constrained MDP is maximally ρ-presentative, this result characterizes the impact on the objective of these constraints and is therefore related to [9]. In our case, the objective is explicit and does not involve *a priori* unknown Lagrange multipliers.

An Alternative Framework for Constructing Portfolios. As we shall see in Section 4.4, our results provide a unifying framework as well-known - possibly constrained - investment strategies maximize an unconstrained objective that is a function of the spectrum $\rho(w)$.

Applications. In Section 5.1, we give a theoretical application of our results on constrained portfolios by extending the "Core Properties" of [7] to the constrained case. In Section 5.3, we perform a numerical experiment where we consider more than 2000 US funds with unknown composition to pinpoint those that qualify for being maximally ρ-presentative. Doing so we also derive a formula to compute the realized diversification of a fund with unknown composition, using time-series only.

1.2 Assumptions and Notations

Assumption. In this paper, we assume that the covariance matrix of the assets Σ is positive-definite. This yields a clear presentation at the cost of a slight loss of generality. To see this, observe that, using the limiting case of the Cauchy-Schwarz inequality, this hypothesis is sufficient to prove:

Proposition 1.1. *Two portfolios are identical up to leverage if and only if they are perfectly correlated.*

If Σ is only positive semi-definite, the proposition does not hold. However, the statements where we prove that portfolios are identical could be reformulated by claiming that they are perfectly correlated. As a result, in several places one can weaken our assumption without impacting significantly the assertions (see Remark 4.17 for a detailed discussion). Note also that we do not assume in this paper that Σ has nonnegative entries. This would have shortened some of our proofs (for instance in Sections 4.2 or 4.3) but would be less relevant to covariances observed in broad financial markets. Lastly, in this paper, we do not discuss how Σ is computed in practice and the data that are used to do so.

Notations. We consider a universe of $n \geq 2$ assets and denote Σ, C, σ, their covariance matrix, correlation matrix, and volatilities vector. These matrices are related by $\Sigma = D(\sigma)CD(\sigma)$, where $D(\sigma)$ is the diagonal matrix with σ as a diagonal. Denoting $\langle \cdot, \cdot \rangle$ the Euclidean inner-product in $\mathbb{R}^n$, the nonnegative $\sigma_\Sigma(w) := \langle \Sigma w, w \rangle^{\frac{1}{2}}$ denotes the *volatility of a portfolio* with weights $w \in \mathbb{R}^n$ and $\|w\|_1 = \sum_{i=1}^n |w_i|$ its *leverage.* Given w with $\sigma_\Sigma(w) > 0$, its *Diversification Ratio* and its *correlation to a portfolio* x with $\sigma_\Sigma(x) > 0$ are defined by

$$DR_\Sigma(w) := \frac{\langle w, \sigma \rangle}{\sigma_\Sigma(w)} \quad \text{and} \quad \varrho_\Sigma(w, x) := \frac{\langle \Sigma w, x \rangle}{\sigma_\Sigma(w)\sigma_\Sigma(x)}.$$

The subscript indicates that the matrix Σ is used for the calculations, and will be omitted when clear from the context. Let us also introduce the set of *long-short unlevered* portfolios and its *long-only unlevered* version:

$$\Pi := \{w \in \mathbb{R}^n \ / \ \|w\|_1 = 1\} \quad \text{and} \quad \Pi^+ := \{w \in \Pi \ / \ \forall i \in \{1, \cdots, n\}, \ w_i \geq 0\}.$$

It is important to note that, *up to leverage, any non-zero long-short portfolio is represented within* Π.

To simplify our calculations, we shall denote $\odot$ (resp. $\oslash$) the entry-wise multiplication (resp. division) between matrices. Whenever we write that a matrix $\Sigma \succ 0$ (resp. $\Sigma \succeq 0$), we mean that it is positive definite (resp. positive semi-definite) whereas when two vectors $x, y \in \mathbb{R}^n$ are such that $x \succ y$ (resp. $x \succeq y$), it means that $\forall i, \ x_i > y_i$ (resp. $\forall i, \ x_i \geq y_i$). For any $x \in \mathbb{R}^n$, we shall denote $x_{(i)}$ the i^{th} order statistic of x, that is defined by the redordering $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$. Alternatively we shall denote $x^\uparrow$ (resp. $x^\downarrow$) the vector that contains the elements of x sorted in non-increasing (resp. non-decreasing) order. Talking about orderings, a function $f : \mathbb{R}^n \to \mathbb{R}$ is *increasing* or *order preserving* if $x \succ y$ implies that $f(x) > f(y)$ for any $x, y \in \mathbb{R}^n$. Such a function, if continuous, is also *non-decreasing* since whenever $x \succeq y$, one has $f(x) \geq f(y)$ for any $x, y \in \mathbb{R}^n$.

We quickly review the portfolios we consider in this paper starting with the EW and the EVW defined by $w_{ew} = \mathbf{1}/n$ and $w_{evw} = \frac{\mathbf{1} \oslash \sigma}{\langle \mathbf{1}, \mathbf{1} \oslash \sigma \rangle}$. The MV w_{mv} minimizes σ_Σ over Π^+ and the ERC solves for $w_{erc} \odot (\Sigma w_{erc}) = n^{-1}\sigma^2(w_{erc})\mathbf{1}$ in Π^+. The long-only MDP w^* maximizes DR_Σ over Π^+. Abusing notations, we call long-short "MDP" the portfolio $\bar{w} := \Sigma^{-1}\sigma / \|\Sigma^{-1}\sigma\|_1$ that maximizes DR_Σ over Π and we always refer to the long-only portfolio when using MDP alone. In addition, we consider the market capitalization weighted portfolio denoted MKT, and a long-short portfolio called PCA and denoted w_{pca} defined as an eigenvector of Σ. We refer to the aforementioned literature for discussions on the existence, uniqueness and other properties of these portfolios.

2 A New Portfolio Representation via Correlations

A key aspect of this paper is the use of an alternative portfolio representation that takes into account *the exposure of a portfolio to all assets* of the investment universe. A candidate for such a representation is the concept of correlation spectrum that we present in this section.

2.1 Definition and Key Property of the Correlation Spectrum

Definition 2.1. The *correlation spectrum* of a portfolio with weights $w \in \mathbb{R}^n \setminus \{0\}$ is the vector $\rho_\Sigma(w) \in \mathbb{R}^n$ such that for any index $i \in \{1, \cdots, n\}$

$$\rho_\Sigma(w)_i := \varrho_\Sigma(w, e_i)$$

where $e_i \in \Pi^+$ is the single-asset portfolio long asset i.

In other words, $\rho_\Sigma(w) = \sigma_\Sigma(w)^{-1}(\Sigma w) \oslash \sigma$. Note that specializing $\Sigma = C$, one has $\rho_C(w) = \sigma_C(w)^{-1}Cw$. The subscripts Σ or C will be omitted whenever clear from the context.

The correlation spectrum *alone* allows to compare the signed exposures of a given portfolio to each asset in the universe. Consider for example a portfolio that has a positive correlation to asset a that is twice that to asset b: a positive one standard deviation return of either asset can be expected to result in a positive portfolio return that is twice as large for asset a than asset b. Note that another measure of exposure, namely the *marginal risk contribution* (see also [16]), will be briefly considered in Section 4.3.2.

Finally, we show that, given a fixed leverage it is equivalent to represent a long-short portfolio by its weights or by its correlation spectrum:

Proposition 2.2. *The mapping $w \in \Pi \mapsto \rho(w) \in \mathcal{E} := \{z \in \mathbb{R}^n, \langle C^{-1}z, z\rangle = 1\}$ is bijective.*

Proof. For $w \in \Pi$, $\langle C^{-1}\rho(w), \rho(w)\rangle = \sigma(w)^{-2}\langle \Sigma^{-1}\Sigma w, \Sigma w\rangle = 1$. Furthermore, given $z \in \mathcal{E}$, we define $\rho^{-1}(z) := \Sigma^{-1}(z \odot \sigma)/\|\Sigma^{-1}(z \odot \sigma)\|_1 \in \Pi$ and verify readily that $\rho \circ \rho^{-1} = \rho^{-1} \circ \rho = I$. □

Example 2.3. To illustrate our definition, we pick the MSCI USA universe and plot in Figure 1 the *independently* sorted vectors $\rho(w)^{\downarrow}$ associated to the EVW, MV, ERC, MDP, long-short MDP and MKT portfolios.

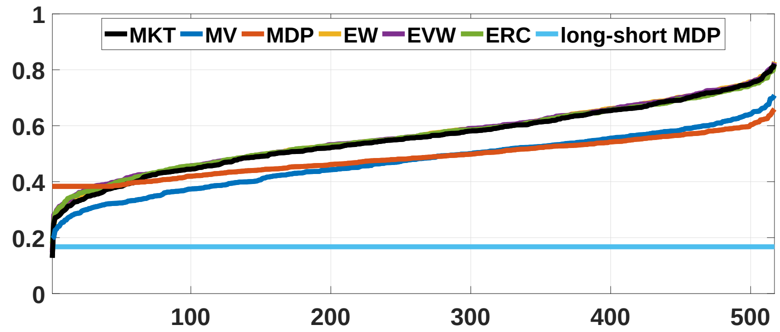

Figure 1: Correlation spectra sorted *independently* for the MDP, long-short MDP, MV, ERC, EVW and MKT portfolio in the MSCI USA over 01/2014-03/2017. The flat region for the MDP spectrum was mentioned in the First Core Property of [7]. Indeed, the MDP is more correlated to the stocks it does not hold than to those it holds and it has the same correlation to the latter ones (see Section 5.1 in this paper for a generalization).

2.2 Other Properties of the Correlation Spectrum

The following proposition contains a composition formula that gives the expression of the spectrum of the convex combination of two long-short portfolios as a function of their individual spectra:

Proposition 2.4. *Take two different $w_0, w_1 \in \Pi$ and $\theta \in (0,1)$ with $w_\theta := \theta w_1 + (1-\theta)w_0 \in \mathbb{R}^n \setminus \{0\}$. Then,*

$$\rho(w_\theta) = d_\theta(\mu_\theta \rho(w_1) + (1-\mu_\theta)\rho(w_0))$$

with $d_\theta = \frac{\theta\sigma(w_1)+(1-\theta)\sigma(w_0)}{\sigma(w_\theta)} > 1$ and $\mu_\theta = \frac{\theta\sigma(w_1)}{\theta\sigma(w_1)+(1-\theta)\sigma(w_0)} \geq 0$.

Proof. As $w_\theta \neq 0$, $\rho(w_\theta) = \frac{1}{\sigma(w_\theta)}\Sigma(\theta w_1 + (1-\theta)w_0) \oslash \sigma = \frac{\theta\sigma(w_1)}{\sigma(w_\theta)}\rho(w_1) + \frac{(1-\theta)\sigma(w_0)}{\sigma(w_\theta)}\rho(w_0)$. As $w_1 \neq w_0$, $d_\theta > 1$ as σ_Σ is strictly convex. □

This result is reminiscent of the diversification axiom in [2] which states that for a coherent risk measure, the risk associated with a weighted combination of assets is no larger than the weighted combination of the individual risks of the assets. Indeed, the scaling d_θ that appears in the above formula measures exactly such an effect. A version of this proposition for an arbitrary number of portfolios is provided in the Appendix.

In the rest of the paper, we shall use the mapping $\phi : \Pi^+ \to \Pi^+$ defined by $\phi(w) := \frac{1}{\langle w, \sigma\rangle} w \odot \sigma$ (and already considered in [6]). It is a bijection and, denoting $x = \phi(w)$, its inverse is given by $w = \phi^{-1}(x) := \frac{1}{\langle x, \mathbf{1} \oslash \sigma\rangle} x \oslash \sigma$. ϕ is helpful as it allows treating assets as if they had identical volatilities. Its use leads to relations that simplify many calculations and that we gather in the following proposition. Its proof is deferred to the Appendix.

Proposition 2.5. *The function ϕ is a bijection from $\Pi^+ \to \Pi^+$ and given $w_1,\ w_2 \in \Pi^+$,* $\rho_\Sigma(w_1) = \rho_C(x_1)\,,\ \varrho_\Sigma(w_1,w_2) = \varrho_C(x_1,x_2),\ \sigma_\Sigma(w_1) = \langle \mathbf{1}, x_1 \oslash \sigma\rangle^{-1}\sigma_C(x_1)$ *and* $DR_\Sigma(w_1) = \sigma_C(x_1)^{-1} \geq 1$.

3 Notions of Representativity and ρ-presentativity

The capitalization weighted index is usually regarded as representative of its investment universe, and has by definition a positive weight on each asset. This consideration leads to:

Definition 3.1. A portfolio $w \in \mathbb{R}^n$ is *representative* if $w \succ 0$.

This definition has some limitations as the weight of an asset in a portfolio may not accurately measure the exposure of the portfolio to that asset. Therefore, to compare the exposure of a portfolio to several stocks we may use a measure that relies on correlations, prompting the following definition:

Definition 3.2. A portfolio $w \in \mathbb{R}^n \setminus \{0\}$, is *$\rho$-presentative* if $\rho(w) \succ 0$.

This definition leaves the way open to long-short portfolios as such a portfolio may be ρ-presentative. Let us now give few examples of ρ-presentative portfolios:

Proposition 3.3. *The long-only ERC, MV and MDP are ρ-presentative. The EW, EVW, the market capitalization weighted index MKT and a PCA portfolio are not necessarily ρ-presentative. The long-short Max Sharpe portfolio $\Sigma^{-1}\mu$ is ρ-presentative if and only if the excess expected returns $\mu \succ 0$.*

Proof. Let us prove that the MDP is ρ-presentative. To do so let us first establish that

$$\underset{w\in\Pi^+}{\operatorname{argmax}}\, DR\,(w) = \phi^{-1}\left(\underset{x\in\Pi^+}{\operatorname{argmin}}\,\sigma_C\,(x)\right).$$

Both $w \mapsto DR\,(w)$ and $w \mapsto \rho(w)$ are well-defined on Π^+. The continuity over the compact Π^+ of $w \mapsto DR\,(w)$ and $x \mapsto \sigma_C\,(x)$ shows that there exist elements in Π^+ maximizing the former and minimizing the latter. Our claim follows from Proposition 2.5 that implies $DR(w) = \sigma_C(x)^{-1}$ hence maximizing DR amounts to minimize σ_C. As $C \succ 0$, $x^* = \phi(w^*)$ is unique and the same holds for w^*. Applying the KKT theorem (cf. [4, 15]) to $\min_{x\in\Pi^+}\sigma_C\,(x)$ shows that the solution x^* solves $Cx^* = \sigma_C^2\,(x^*)\,\mathbf{1} + \lambda,$ with $\lambda \odot x^* = 0$ and $\lambda \succeq 0$, hence $\rho_C\,(x^*) = \sigma_C\,(x^*)\,\mathbf{1} + \frac{\lambda}{\sigma_C(x^*)}$. So $\min\rho_C\,(x^*) = \sigma_C\,(x^*)$ as $x^* \in \Pi^+$ has a positive entry associated to a zero entry of λ. Finally, by Proposition 2.5, $\rho_\Sigma\,(w^*) = \rho_C\,(x^*)$ and $\sigma_C(x^*) = DR(w^*)^{-1}$. To sum up,

$$\min\rho_\Sigma\,(w^*) = \min\rho_C\,(x^*) = \sigma_C\,(x^*) = DR\,(w^*)^{-1} > 0. \tag{3.1}$$

In particular, the MDP w^* satisfies $\rho\,(w^*) \succeq \frac{\mathbf{1}}{DR(w^*)} \succ 0$.

Similarly, by the MV first order condition, $\Sigma w_{mv} \succeq \sigma^2\,(w_{mv})\,\mathbf{1}$, hence, $\rho\,(w_{mv}) \succeq \sigma\,(w_{mv}) \oslash \sigma \succ 0$. The long-only ERC solves $w_{erc} \odot (\Sigma w_{erc}) = n^{-1}\sigma^2\,(w_{erc})\mathbf{1}$, hence, $\rho\,(w_{erc}) = n^{-1}\sigma\,(w_{erc}) \oslash (w_{erc} \odot \sigma) \succ 0$. Note that, this portfolio exhibits a nice feature if ρ-presentativity is the goal: the lower the correlation to an asset, the higher its weight. Taking the EW or EVW of a large collection of highly correlated assets to which is added another asset sufficiently negatively correlated to the others proves that these portfolios are not ρ-presentative. The same argument holds in theory for the MKT portfolio. Lastly, observe that $\forall i \in \{1, \cdots, n\}$, $\operatorname{sgn}((\rho(w_{pca}))_i) = \operatorname{sgn}((w_{pca})_i)$ so w_{pca} is not necessarily ρ-presentative. □

The classes of representative and ρ-presentative portfolios intersect as the ERC lies in both. However, these two classes are not included in one another: as there exist representative portfolios that are not ρ-presentative, there are ρ-presentative portfolios that are not necessarily representative. The MDP is such a portfolio (cf. for instance Figure 1). Lastly, the Max Sharpe portfolio is an example of a portfolio that is not necessarily long-only but that may happen to be ρ-presentative.

Let us pursue with a fundamental property of ρ-presentative portfolios:

Lemma 3.4. *Given a ρ-presentative portfolio w, the (not necessarily unique) least correlated long-only portfolio to w is an asset. Actually for $w \in \Pi$ such that $\rho(w) \succeq 0$,*

$$\min_{\theta\in\Pi^+}\varrho\,(w,\theta) = \min\rho\,(w)\,. \tag{3.2}$$

This is based on the fact that $\forall\,(w,\theta) \in (\mathbb{R}^n \setminus \{0\}, \Pi^+)$, $\varrho\,(w,\theta) = DR\,(\theta)\,\langle\phi\,(\theta), \rho\,(w)\rangle$.

Proof. The last identity follows from the generalized version of Proposition 2.4 that is in the Appendix but we can also give a short and direct proof as $\forall (w, \theta) \in (\mathbb{R}^n \setminus \{0\}, \Pi^+)$,

$$\varrho(w, \theta) = \frac{\langle w, \Sigma\theta \rangle}{\sigma(w)\,\sigma(\theta)} = \frac{\langle \theta \odot \sigma, \rho(w) \rangle}{\sigma(\theta)} = \frac{\langle \theta, \sigma \rangle}{\sigma(\theta)} \langle \phi(\theta), \rho(w) \rangle = DR(\theta) \langle \phi(\theta), \rho(w) \rangle .$$

To prove the identity (3.2), observe that the infimum over θ is always smaller than the right-hand side so we just need to focus on the reverse inequality. As $\phi(\theta) \in \Pi^+$, $\forall z \in \mathbb{R}^n$, $\langle \phi(\theta), z \rangle \geq \min(z)$, and so

$$\varrho(w, \theta) = DR(\theta) \langle \phi(\theta), \rho(w) \rangle \geq DR(\theta) \min \rho(w) \geq \min \rho(w) \tag{3.3}$$

where we used $DR(\theta) \geq 1$ and our assumption that guaranties that $\min \rho(w) \geq 0$. We conclude the proof of the identity by taking the minimum with respect to $\theta \in \Pi^+$, which exists by continuity of $\theta \mapsto \varrho(w, \theta)$ on Π^+.

In case $\rho(w) \succ 0$, assume that the min over θ is attained by $\theta^* \in \Pi^+$, then combining (3.3) and the fact that $\varrho(w, \theta^*) = \min \rho(w)$, one has $DR(\theta^*) = 1$. As σ_Σ is strictly convex, this is possible only if θ^* is an asset. □

The lemma implies that, whenever all entries of Σ are positive, the least correlated long-only portfolio to another long-only portfolio is an asset. However, in general one cannot drop the assumption $\rho(w) \succeq 0$ as one can build a counter-example with a matrix that has negative entries and where we can verify that

$$\min_{\theta \in \Pi^+} \varrho(w, \theta) < \min \rho(w)$$

for some portfolios w that are therefore not ρ-presentative. See Figure 5 on page 21 for such a counter-example that cannot occur in the classical Euclidean setting.

A consequence of Lemma 3.4 is derived from its combination with the identity (3.1):

Proposition 3.5. *A long-only portfolio is positively correlated to at least one asset, as, denoting* w^* *the MDP,*

$$\min_{w \in \Pi^+} \max \rho(w) \geq [\min \rho(w^*)]^2 = [DR(w^*)]^{-2} > 0.$$

Proof. Given $w \in \mathbb{R}^n \setminus \{0\}$ and considering ϕ as defined before Proposition 2.5, $\phi(w^*) \in \Pi^+$ which implies that

$$\max \rho(w) \geq \langle \phi(w^*), \rho(w) \rangle = DR(w^*)^{-1} \varrho(w, w^*) = \min \rho(w^*)\, \varrho(w, w^*),$$

where we applied the last part of Lemma 3.4. Then we take on both ends the minimum (which exists by continuity of $\max(\rho(\cdot))$ and $\rho(\cdot, w^*)$ on the compact Π^+) and as $\rho(w^*) \succ 0$, we can apply (3.2) in Lemma 3.4:

$$\min_{w \in \Pi^+} \max \rho(w) \geq \min \rho(w^*) \min_{w \in \Pi^+} \varrho(w, w^*) = [\min \rho(w^*)]^2 > 0. \qquad \square$$

As a weak converse of this result, observe that a ρ-presentative porfolio w cannot be short-only as by (3.1) one has $0 < \langle w, \Sigma w^* \rangle$ and $\Sigma w^* \succ 0$. This direction will be explored further in the next section.

4 Maximally ρ-presentative Portfolios

As we have seen in the previous section, it is possible to build portfolios that are ρ-presentative *i.e.* such that they have a positive exposure to all assets. In this section we introduce a complementary notion by considering portfolios that maximize their overall exposure to all assets.

4.1 Definition and Properties of Maximally ρ-presentative Portfolios

Definition 4.1. A portfolio $w_f \in \mathbb{R}^n \setminus \{0\}$ is *maximally ρ-presentative* if there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ that is increasing, symmetric and concave such that

$$w_f \in \underset{\mathbb{R}^n \setminus \{0\}}{\operatorname{argmax}} f \circ \rho. \tag{4.1}$$

We shall denote $\mathcal{R}$ the set of all unlevered maximally ρ-presentative portfolios.

A maximally ρ-presentative portfolio maximizes its exposures to all assets through an aggregate view offered by f which measures how ρ-presentative a portfolio is as a whole, given its exposures. Specifically:

(i) f is increasing to advantage a portfolio that is more ρ-presentative than another. In other words, if $\rho(w) \succ \rho(y)$ then $f \circ \rho(w) > f \circ \rho(y)$. This assumption excludes for instance $f = \|\cdot\|_2$.

(ii) f is concave which is consistent with the property of $\rho(w_\theta)$ in Proposition 2.4. Furthermore, we shall see that this assumption is key to prove that for fixed f, there is a unique maximally ρ-presentative portfolio. As a counter-example, for $f(x) = \sum_{i=1}^n x_i^3$ and $\Sigma = I$, optima of (4.1) are single asset portfolios.

(iii) f is symmetric *i.e.* invariant under a permutation of coordinates as there is *a priori* no reason for it to change if we permute the exposures. This excludes $f = \langle \cdot, \theta \rangle$ with $\theta \in \Pi^+ \setminus \{n^{-1}\mathbf{1}\}$.

The examples we just gave will be further discussed in Section 4.4 and we shall see in the sequel how the concepts of ρ-presentative and maximally ρ-presentative portfolios compare to each other.

Let us now establish the existence and uniqueness of a such a portfolio for a given f:

Proposition 4.2. *For a concave increasing f, the maximum in (4.1) is reached by a unique unlevered portfolio.*

Proof. We denote $\|\cdot\|$ the norm associated to the inner-product defined by C^{-1}. As $\rho_\Sigma : \Pi \to \mathcal{E} = \{z, \|z\| = 1\}$ is a bijection, one has $\sup_\Pi f \circ \rho = \sup_\mathcal{E} f$ with the left problem having a unique maximum iff the same is true for the right one so we may focus on the latter one.

Existence: as in finite dimension, any concave function is continuous in the interior of its domain (see the monograph [15, Theorem 10.1]), f attains its supremum m^* on the compact ball $\mathcal{E}$.

Uniqueness: assuming the contrary, there are $z_1 \neq z_2$ such that $\|z_1\| = \|z_2\| = 1$ and $f(z_1) = f(z_2) = m^*$. Then considering a strict convex combination z_θ of z_1 and z_2, $\|z_\theta\| < 1$ by strict convexity of the norm whereas by concavity of f, $f(z_\theta) \geq m^*$. Now as $\lambda \mapsto \|z_\theta + \lambda \mathbf{1}\|$ is continuous on $[0, +\infty)$ and tends to $+\infty$ when $\lambda \to +\infty$, by the intermediate value theorem $\exists \lambda^* \in (0, +\infty)$ such that $\|z_\theta + \lambda^* \mathbf{1}\| = 1$. On the other hand, as f is increasing, $f(z_\theta + \lambda^* \mathbf{1}) > f(z_\theta) \geq m^*$, hence a contradiction with the definition of m^*. □

The following result and ensuing theorem show that long-only portfolios have a special role amongst long-short portfolios seeking to maximize their exposure to all assets:

Lemma 4.3. *For any $y \in \Pi \setminus \Pi^+$, there exists $w \in \Pi^+$ such that $\rho(w) \succ \rho(y)$. However this cannot hold for long-only portfolios. Indeed, if $w \in \Pi$, $y \in \Pi^+$ and $\rho(w) \succeq \rho(y)$ then $w = y$.*

Proof. To prove the first statement, consider the convex problem $\min_\mathcal{C} \sigma_\Sigma^2$ with $\mathcal{C} = \{z \in \mathbb{R}^n \;/\; z \succeq 0, \Sigma z \succeq \Sigma y\}$. It is feasible as we may always consider a rescaled enough long-only ρ-presentative portfolio and admits a unique solution v since its objective is strictly convex and the constraints are linear. Without loss of generality, one can assume that $v \neq 0$ as otherwise $\Sigma y \preceq 0$ and any long-only ρ-presentative satisfies our first statement.

Denoting $\lambda, \mu \succeq 0$ the Lagrange multipliers associated to the constraints $v \succeq 0$ and $\Sigma(v - y) \succeq 0$, the solution v solves the following KKT conditions: $\Sigma v = \lambda + \Sigma \mu$, $\lambda \odot v = 0$ and $\mu \odot (\Sigma(v - y)) = 0$. The first two conditions imply that $\sigma^2(v) = \langle \Sigma v, \mu \rangle$ and $\langle \Sigma v, \mu \rangle = \langle \lambda, \mu \rangle + \sigma^2(\mu) \geq \sigma^2(\mu)$ while the last one implies $\langle \Sigma v, \mu \rangle = \langle \Sigma y, \mu \rangle$. Then, as a result of the Cauchy-Schwarz inequality,

$$\sigma^2(v) = \langle \Sigma v, \mu \rangle = \langle \Sigma y, \mu \rangle \leq \sigma(y)\sigma(\mu) \leq \sigma(y)\langle \Sigma v, \mu \rangle^{\frac{1}{2}} = \sigma(y)\sigma(v). \tag{4.2}$$

Since $v \neq 0$, we have $\sigma(v) > 0$ and therefore $\sigma(v) \leq \sigma(y)$.

Let us prove that $\sigma(v) = \sigma(y)$ cannot occur. If the identity holds then all inequalities in (4.2) are equalities hence $\langle \lambda, \mu \rangle = 0$ and $\langle \Sigma y, \mu \rangle = \sigma(y)\sigma(\mu)$. Then by the limiting case of the Cauchy-Schwarz inequality, there exists $\gamma \in \mathbb{R}$ such that $y = \gamma \mu$. Combining this observation and the fact that $\langle \lambda, \mu \rangle = 0$ with the first KKT condition yields $\langle \Sigma v, y \rangle = \langle \lambda, y \rangle + \langle \Sigma \mu, y \rangle = \gamma \langle \lambda, \mu \rangle + \langle \Sigma y, \mu \rangle = \langle \Sigma y, \mu \rangle$. Put together with (4.2) this implies that $\langle \Sigma v, y \rangle = \sigma(v)\sigma(y)$ *i.e.* $\varrho(v, y) = 1$ and thus $y = v \in \Pi^+$. This contradicts our assumption on y.

Therefore one necessarily has $\sigma(v) < \sigma(y)$. Consider a long-only ρ-presentative that we rescale enough so that $u \in \mathcal{C}$, $\Sigma u \succ \Sigma v \succeq \Sigma y$, and $\sigma(u) > \sigma(y) > \sigma(v)$. By continuity of σ_Σ on $\mathcal{C}$, there exists by the intermediate value theorem a strict convex combination $w = \alpha u + (1 - \alpha) v \in \mathcal{C}$ with $\alpha \in (0, 1)$ such that $\sigma(w) = \sigma(y)$. Then, as $\Sigma w = \alpha \Sigma u + (1 - \alpha) \Sigma v \succ \Sigma y$, one has $\frac{1}{\sigma(w)} \Sigma w = \frac{1}{\sigma(y)} \Sigma w \succ \frac{1}{\sigma(y)} \Sigma y$ hence $\rho(w) \succ \rho(y)$ (the proof is constructive as α is the root of a quadratic equation that one can readily compute).

To prove the second statement, rescaling y, we may consider that $y \succeq 0$ and $\sigma(y) = \sigma(w)$. Then the assumption implies $\Sigma w \succeq \Sigma y$. Taking the inner-product with y, $\langle \Sigma w, y \rangle \geq \sigma^2(y) = \sigma(w)\sigma(y)$ hence $w = y$. □

Before getting further we recall that for any $v \in \mathbb{R}^n$ we denote $v^\uparrow$ (resp. $v^\downarrow$) the vector that contains the elements of v sorted in non-increasing (resp. non-decreasing) order and that the bijective $\phi : \Pi^+ \to \Pi^+$ is defined by $\phi(w) := \frac{1}{\langle w, \sigma \rangle} w \odot \sigma$. Moreover we define $(\Pi^+)^\uparrow := \{w \in \Pi^+ \ / \ 0 \leq w_n \leq w_{n-1} \leq \cdots \leq w_1 \leq 1\}$. Equipped with these notations and Lemma 4.3 we are ready to prove the main result of this section:

Theorem 4.4. *Maximally ρ-presentative portfolios are exactly the portfolios $w \in \Pi^+$ that satisfy*

$$\langle \phi(w)^\uparrow, \rho(w)^\downarrow \rangle = \langle \phi(w), \rho(w) \rangle. \tag{4.3}$$

In other words, as $\langle \phi(w), \rho(w) \rangle = DR(w)^{-1}$,

$$\mathcal{R} = \underset{w \in \Pi^+}{\operatorname{argmax}} \left(\langle \phi(w)^\uparrow, \rho(w)^\downarrow \rangle DR(w) \right).$$

Moreover, the mapping

$$w \in (\Pi^+)^\uparrow \mapsto \underset{y \in \Pi}{\operatorname{argmax}} \langle w, \rho(y)^\downarrow \rangle \in \mathcal{R} \tag{4.4}$$

is well-defined and surjective onto the set of maximally ρ-presentative portfolios.

Representing maximally ρ-presentative portfolios w by their volatility adjusted weights $\phi(w) \in \Pi^+$, their $(n-1)$-dimensional Lebesgue measure λ_{n-1} is such that

$$\lambda_{n-1}(\phi(\mathcal{R})) \leq \frac{\lambda_{n-1}(\Pi^+)}{n!}.$$

And lastly, $\mathcal{R}$ is a finite union of polytopes.

Proof. Let us start by proving that maximally ρ-presentative portfolios are long-only. Consider $y_f \in \operatorname{argmax}_\Pi f \circ \rho$ for some f that is continuous and increasing. Then $y_f \in \Pi^+$ as otherwise, by Lemma 4.3, $\exists w_f \in \Pi^+$ such that $\rho(w_f) \succ \rho(y_f)$, hence $f \circ \rho(w_f) > f \circ \rho(y_f)$ and y_f is not optimal.

Now, remark that by the last identity in Lemma 3.4

$$\forall \theta \in \Pi^+, \ \max_{w \in \Pi} \langle \phi(\theta), \rho(w) \rangle = \langle \phi(\theta), \rho(\theta) \rangle = DR^{-1}(\theta) > 0 \tag{4.5}$$

which means that $\phi(\theta)$ is an outer normal to the ellipsoid $\mathcal{E}$ at $\rho(\theta)$.

We pursue by proving that any maximally ρ-presentative w_f also satisfies (4.3) and we refer to Figure 2 for the geometric intuition behind the argument. Considering $\mathfrak{S}_n$ the group of permutations of $\{1, \cdots, n\}$ let us first note that for any $w \in \Pi^+$,

$$\langle \phi(w)^\uparrow, \rho(w)^\downarrow \rangle = \min_{p \in \mathfrak{S}_n} \langle \phi(w), p \circ \rho(w) \rangle.$$

Assuming that (4.3) does not hold, there exists $p \in \mathfrak{S}_n$ with $\langle \phi(w_f), p \circ \rho(w_f) \rangle < \langle \phi(w_f), \rho(w_f) \rangle$. As $\phi(w_f)$ is an outer normal to the ellipsoid $\mathcal{E}$ at $\rho(w_f)$, there exists a strict convex combination z_θ of $p \circ \rho(w_f)$ and $\rho(w_f)$ that lies in the interior of the domain enclosed by $\mathcal{E}$. We may then conclude as in the proof of the uniqueness in Proposition 4.2. Indeed, as f is concave and symmetric $f(z_\theta) \geq f \circ \rho(w_f)$. Then by the intermediate value theorem $\exists \lambda^* \in (0, +\infty)$ such that $z_\theta + \lambda^* \mathbf{1} \in \mathcal{E}$. Thus $\exists y \in \Pi$ such that $\rho(y) = z_\theta + \lambda^* \mathbf{1}$. As f is increasing $f \circ \rho(y) > f(z_\theta) \geq f \circ \rho(w_f)$ contradicting the optimality of w_f. Thus w_f satisfies (4.3).

Conversely, given $\theta \in \Pi^+$ we introduce the function $f_\theta : z \mapsto \min_{p \in \mathfrak{S}_n} \langle \phi(\theta), p(z) \rangle$. This mapping as well as (4.4) are well-defined as the objectives are continuous over the compact Π. Moreover f_θ is increasing, symmetric and concave on $\mathbb{R}^n$ and $f_\theta(z) = \langle \phi(\theta)^\uparrow, z^\downarrow \rangle$. So if we take $\theta \in \Pi^+$ that satisfies (4.3), then

$$\langle \phi(\theta), \rho(\theta) \rangle = f_\theta \circ \rho(\theta) \leq \max_{w \in \Pi} f_\theta \circ \rho(w) = \max_{w \in \Pi} \min_{p \in \mathfrak{S}_n} \langle \phi(\theta), p \circ \rho(w) \rangle \leq \max_{w \in \Pi} \langle \phi(\theta), \rho(w) \rangle = \langle \phi(\theta), \rho(\theta) \rangle \tag{4.6}$$

where in the two last steps we took $p = Id$ and used (4.5). Thus θ maximizes $f_\theta \circ \rho$ hence, $\theta \in \mathcal{R}$.

By the previous analysis, for any $w \in \mathcal{R}$, $\exists x := \phi(w)^\uparrow \in (\Pi^+)^\uparrow$ such that $w = \operatorname{argmax}_{y \in \Pi} \langle x, \rho(y)^\downarrow \rangle$ *i.e.* (4.4) is surjective. In addition, by the partition principle, there exists an injection $\mathcal{R} \hookrightarrow (\Pi^+)^\uparrow$.

Given $p \in \mathfrak{S}_n$, we denote $\Delta_p := \{w \in \Pi^+ \ / \ p \circ \phi(w) = \phi(w)^\uparrow, \ p \circ \rho(w) = \rho(w)^\downarrow\}$ the portfolios whose volatility weighted weights and spectra are ordered in opposite directions by the same permutation p. This set is a polytope and we observe that if not empty $\mathcal{R} \cap \{w \in \Pi^+ \ / \ p(\phi(w)) = \phi(w)^\uparrow\} = \Delta_p$. Thus $\mathcal{R}$ is a finite union of polytopes as $\mathcal{R} = \bigcup_{p \in \mathfrak{S}_n} \Delta_p$.

In the sequel, through the use of ϕ, we can assume that $\sigma = \mathbf{1}$. First, observe that for $w \in \mathcal{R}$ and any permutation p such that $p(w) \neq w$, $p(w) \notin \mathcal{R}$, Indeed, reasoning as in (4.6), we can show that $\langle w, \rho(w)\rangle \leq \langle p(w), \rho(p(w))\rangle$. So, if both $w, p(w) \in \mathcal{R}$, one has $\langle w^\uparrow, \rho(w)^\downarrow\rangle \leq \langle p(w)^\uparrow, \rho(p(w))^\downarrow\rangle$ that is $\max_{\Pi^+} f_w \circ \rho = f_w \circ \rho(w) \leq f_w \circ \rho(p(w))$. As by Proposition 4.2, f_w admits a unique maximum, $p(w) = w$. Note that this also tells us that if $w \in \mathcal{R}$, then for any permutation p such that $p(w) \neq w$, one has $\sigma_\Sigma(w) < \sigma_\Sigma(p(w))$.

We are now ready to prove that the measure of $\mathcal{R}$ is small as compared to that of Π^+. Denoting λ_{n-1} the $(n-1)$-dimensional Lebesgue measure, we remark that by (4.3) - that we have now proven - $\mathcal{R}$ is closed, hence λ_{n-1}-measurable. Now, denoting $\mathcal{N} \subset \Pi^+$, the set of portfolios with each having at least two identical weights, its measure $\lambda_{n-1}(\mathcal{N}) = 0$. Thus, defining $\widetilde{\mathcal{R}} = \mathcal{R} \setminus \mathcal{N}$ the set of maximally ρ-presentative portfolios that have distinct coordinates,

$$\lambda_{n-1}(\mathcal{R}) = \sum_{p \in \mathfrak{S}_n} \lambda_{n-1}\left[\widetilde{\mathcal{R}} \cap \{w \in \Pi^+,\ p(w) = p(w)^\uparrow\}\right] = \sum_{p \in \mathfrak{S}_n} \lambda_{n-1}\left[p\left(\widetilde{\mathcal{R}} \cap \{w \in \Pi^+,\ p(w) = p(w)^\uparrow\}\right)\right]$$

as permutations are isometries. Now as any permutation of $w \in \mathcal{R}$ that is distinct from it is not in $\mathcal{R}$, the measure of $\mathcal{R}$ is equal to the measure of the union of the *disjoint* sets $p(\widetilde{\mathcal{R}} \cap \{w \in \Pi^+,\ p(w) = p(w)^\uparrow\})$ that all belong to $(\Pi^+)^\uparrow$ and is therefore smaller than $\frac{1}{n!}\lambda_{n-1}(\Pi^+)$. □

In fact, this theorem shows that for any maximally ρ-presentative portfolio w, there exists a permutation that sorts its volatility weighted weights $\phi(w)$ in non-decreasing order and its exposures $\rho(w)$ in non-increasing order. Therefore, its exposures form a non-increasing function of its volatility weighted weights. This theorem also shows that maximally ρ-presentative portfolios are rare. Indeed, given n assets, if one drew uniformly the volatility adjusted weights of N long-only portfolios, there is less than $\frac{N}{n!}$ chance to have drawn those of a maximally ρ-presentative portfolio. Finally, the above characterization allows to prove that maximally ρ-presentative portfolios are diversified in the sense that their Diversification Ratio is never less than that of an EVW portfolio. More precisely:

Proposition 4.5. *A maximally ρ-presentative portfolio w_f satisifies the following bounds*

$$0 < \frac{DR(w_{evw})}{\varrho(w_f, w_{evw})} \leq DR(w_f) \leq DR(w^*)\varrho(w_f, w^*), \tag{4.7}$$

where we recall that w^* denotes the MDP.
In terms of the objective f, we have the following tight bounds

$$f(DR(w^*)^{-1}\mathbf{1}) \leq f \circ \rho(w_f) \leq f(DR(w_{evw})^{-1}\mathbf{1}). \tag{4.8}$$

Proof of (4.7): By Proposition 2.5, the last identity in Lemma 3.4 and the characterization (4.3),

$$DR^{-1}(w_f) = \min_{p \in \mathfrak{S}_n} \langle p \circ \phi(w_f), \rho_\Sigma(w_f)\rangle = \min_{p \in \mathfrak{S}_n} \langle p(x_f), \rho_C(x_f)\rangle \leq \langle \rho_C(x_f), n^{-1}\mathbf{1}\rangle = DR^{-1}(w_{evw})\varrho(w_f, w_{evw}).$$

As by Theorem 4.4, $w_f \in \Pi^+$ then $DR(w_f) \leq DR(w^*)\varrho(w_f, w^*)$ by the second core property in [7] (this latter result will be generalized in Proposition 5.2). This finishes the proof of (4.7).
Proof of (4.8): Considering the shift operator $S(x_1, \cdots, x_n) = (x_2, \cdots, x_n, x_1)$, $\forall w \in \mathbb{R}^n \setminus \{0\}$,

$$f \circ \rho(w) = \frac{1}{n}\sum_{k=1}^n f \circ \rho(w) = \frac{1}{n}\sum_{k=1}^n f(S^k \rho(w)) \leq f\left(\frac{1}{n}\sum_{k=1}^n S^k \rho(w)\right) = f(\langle \rho(w), n^{-1}\mathbf{1}\rangle \mathbf{1}) = f(\langle \rho(w), w_{ew}\rangle \mathbf{1})$$

where we invoked the symmetry and the concavity. Then, invoking Lemma 3.4, for any $w \in \mathbb{R}^n \setminus \{0\}$,

$$f \circ \rho(w) \leq f(\langle \rho(w), w_{ew}\rangle \mathbf{1}) = f(\langle \rho(w), \phi(w_{evw})\rangle \mathbf{1}) = f(DR(w_{evw})^{-1}\varrho(w, w_{evw})\mathbf{1}). \tag{4.9}$$

Therefore, as f is increasing, $\max_\Pi f \circ \rho \leq f(DR(w_{evw})^{-1}\mathbf{1})$ whereas on the other hand by (3.1) and (4.9),

$$f(DR(w^*)^{-1}\mathbf{1}) = f(\min \rho(w^*)\mathbf{1}) \leq f \circ \rho(w^*) \leq \max_\Pi f \circ \rho \leq f(\langle \rho(w_f), n^{-1}\mathbf{1}\rangle \mathbf{1}). \tag{4.10}$$

Inequalities (4.9) and (4.10) are illustrated in Figure 2. □

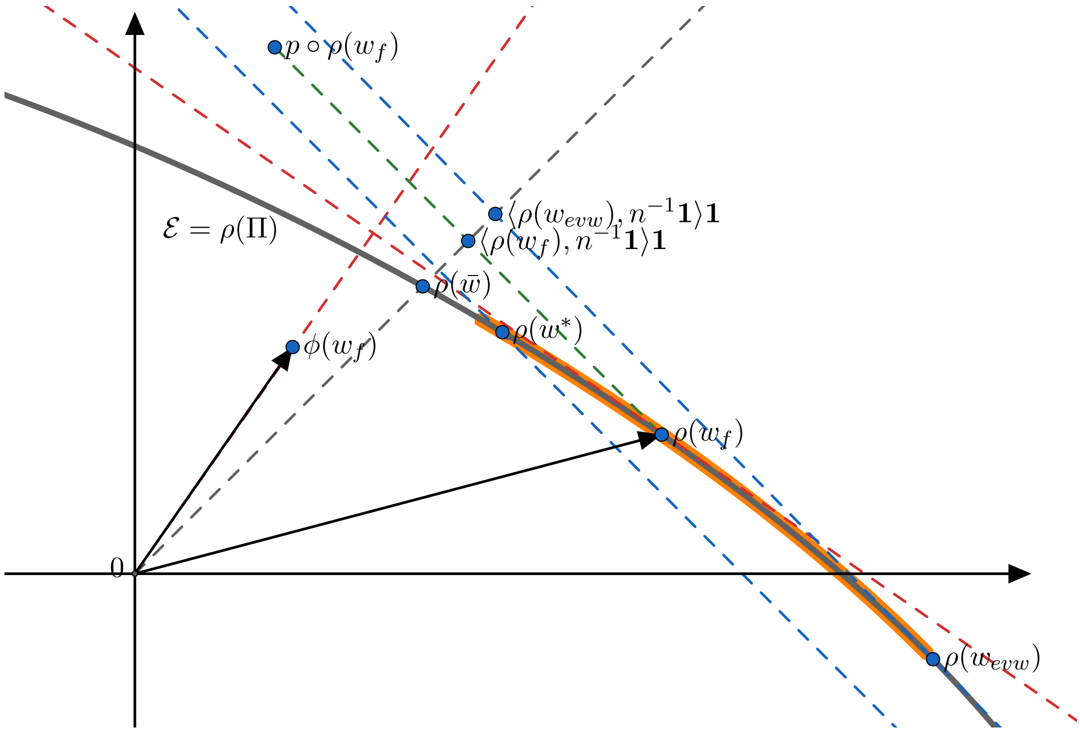

Figure 2: A schematic view of the geometry behind the proofs and results of Proposition 4.2, Theorem 4.4 and Proposition 4.5. The dashed lines in red are orthogonal as $\phi(w_f)$ is an outer normal to the ellipsoid $\mathcal{E}$ at $\rho(w_f)$. The set $\mathcal{R}$ of maximally ρ-presentative portfolios *resides in* the orange region that is the intersection between the surface $\mathcal{E}$ and the strip delimited by the blue hyperplanes. As f is increasing, this comes from (4.9) and (4.10). The green dashed segment is the set of convex combinations of the permutations $p \circ \rho(w_f)$.

Portfolios that are ρ-presentative are exactly those that are positively correlated with all long-only portfolios. In general, a maximally ρ-presentative portfolio is not ρ-presentative. However, we shall see in the following proposition how these two concepts come together:

Proposition - Definition 4.6. *A maximally ρ-presentative portfolio w_f is* weakly ρ-presentative *in the sense that its average exposure is positive. Moreover, we have a bound uniform in f that involves the MDP w^*:*

$$n^{-1}\langle \rho(w_f), \mathbf{1}\rangle \geq DR^{-1}(w_f) \geq \min \rho(w^*) > 0. \tag{4.11}$$

Furthermore maximally ρ-presentative portfolios are positively correlated to a special long-only portfolio, namely the EVW. In particular,

$$\varrho(w_f, w_{evw}) \geq \frac{DR(w_{evw})}{DR(w^*)}. \tag{4.12}$$

Proof. Combining (3.1) and (4.7) we obtain (4.11). Also, as f is increasing, we get $n^{-1}\langle \rho(w_f), \mathbf{1}\rangle \geq \min \rho(w^*)$ directly from (4.10) without this time relying on Theorem 4.4. Inequality (4.7) yields directly (4.12). □

According to (4.7), portfolios reduced to assets - and whose DR equals one - are obviously long-only but never maximally ρ-presentative. In the following proof, we are going to consider another example of a long-only portfolio that is never maximally ρ-presentative, namely

$$w^\sharp \in \underset{w \in \Pi^+}{\operatorname{argmin}}\, \varrho(w, w_{evw}).$$

This portfolio may happen to be different from any asset as soon as the EVW is not ρ-presentative (see Lemma 3.4 and the remark below it). As indicated by the previous theorem, many long-only portfolios are not maximally ρ-presentative. In particular, considering the limiting case of inequality (4.12) with $w^* = w_{evw}$, it appears that there is only one maximally ρ-presentative portfolio which is the EVW/MDP. This particular case occurs if and only if $\mathbf{1}$ is an eigenvector of the correlation matrix, as shown by the KKT conditions given by the MDP problem. Nonetheless, in general, $\mathcal{R}$ is not a singleton but it is not large either as in the following proposition we show that there are large open regions of Π^+ with no maximally ρ-presentative portfolios.

Proposition 4.7. *The set of maximally ρ-presentative portfolios satisfies the following inclusions:*

$$\mathcal{R} \subset \mathcal{F} := \left\{ w \in \Pi^+,\ \varrho(w, w_{evw}) \geq \frac{DR(w_{evw})}{DR(w)} \right\} \subset \tilde{\mathcal{F}} := \left\{ w \in \Pi^+,\ \varrho(w, w_{evw}) \geq \frac{DR(w_{evw})}{DR(w^*)} \right\} \subsetneq \Pi^+.$$

Moreover, both $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are closed convex sets. In particular, the tangent hyperplane to $\mathcal{F}$ at w_{evw} separates Π^+ in two sets such that one of them does not contain any maximally ρ-presentative portfolio.

Lastly, the last inequality in the definition of $\tilde{\mathcal{F}}$ can be taken strict if the long-short MDP $\bar{w} \neq w^*$.

Proof. Step 1: The inclusion $\mathcal{R} \subset \mathcal{F}$ follows from (4.7). Denoting $\bar{w} = \Sigma^{-1}\sigma$, we remark that for any $w \in \Pi$,

$$DR(\bar{w})\varrho(\bar{w}, w) = \frac{\langle \sigma, \Sigma^{-1}\sigma \rangle}{\sigma_\Sigma(\Sigma^{-1}\sigma)^2} \frac{\langle \Sigma\Sigma^{-1}\sigma, w \rangle}{\sigma_\Sigma(w)} = \frac{\langle \sigma, \Sigma^{-1}\sigma \rangle}{\langle \Sigma\Sigma^{-1}\sigma, \Sigma^{-1}\sigma \rangle} \frac{\langle \sigma, w \rangle}{\sigma_\Sigma(w)} = 1 \times DR(w). \tag{4.13}$$

Combining this identity with (4.7) yields $\forall w \in \mathcal{R}$, $\varrho(w_{evw}, \bar{w}) \leq \varrho(w, w_{evw})\varrho(w, \bar{w})$ which, letting $\lambda := \langle \Sigma w_{evw}, \bar{w} \rangle$, can be rewritten $\lambda\sigma(w)^2 \leq \langle \Sigma w, w_{evw} \rangle \langle \Sigma w, \bar{w} \rangle = \frac{1}{4}\left(\langle w, \Sigma(\bar{w} - w_{evw}) \rangle^2 + \langle w, \Sigma(\bar{w} + w_{evw}) \rangle^2 \right)$. Let us now remark that $\forall w \in \mathcal{R}$, $\langle w, \Sigma(\bar{w} + w_{evw}) \rangle \geq 0$ as by (4.7), $\langle w, \Sigma w_{evw} \rangle \geq 0$ and that $\forall w \in \mathcal{R}, \langle w, \Sigma\bar{w} \rangle \geq 0$ as this is true in Π^+ by (4.13) and as $\mathcal{R} \subset \Pi^+$ by Theorem 4.4. Then considering the matrix $M := \lambda\Sigma + (\Sigma\frac{(\bar{w} - w_{evw})}{2})(\Sigma\frac{(\bar{w} - w_{evw})}{2})'$ we may rewrite $\mathcal{F} = \{ w \in \Pi^+,\ \|w\|_M \leq \langle w, \Sigma(\frac{w_{evw} + \bar{w}}{2}) \rangle \}$. Note that $\lambda > 0$ once again by (4.13) so $M \succ 0$ and therefore $\mathcal{F}$ is closed and convex as it is the intersection of a closed and non-degenerate hyperbolic cone with the regular simplex Π^+. The rest of the claim follows from the fact that w_{evw} lies on the boundary of $\mathcal{F}$ and in the interior of Π^+.

Step 2: As $\forall w \in \Pi^+$, $DR(w^*) \geq DR(w)$, $\mathcal{F} \subset \tilde{\mathcal{F}}$. To establish $\tilde{\mathcal{F}} \subsetneq \Pi^+$, let us prove that $O_1 := \{ w \succ 0,\ \varrho(w, w_{evw})DR(w^*) < DR(w_{evw}) \}$ is not empty. Considering (4.9) we are tempted to take the minimum on both sides and to do so let $w^\sharp \in \operatorname{argmin}_{w \in \Pi^+} \varrho(w, w_{evw})$ that does not depend on f and that exists as the objective is continuous on the compact Π^+. As $\varrho(w^\sharp, w_{evw}) \leq \min \rho(w_{evw})$, we have by (4.9) $f \circ \rho(w^\sharp) \leq f(DR(w_{evw})^{-1} \min \rho(w_{evw})\mathbf{1})$. If $\min \rho(w_{evw}) > 0$, then as $DR(w_{evw}) > 1$, $f \circ \rho(w^\sharp) < f(\min \rho(w_{evw})\mathbf{1}) \leq f \circ \rho(w_{evw})$. Otherwise, we know there exists a ρ-presentative $u \in \mathbb{R}^n \backslash \{0\}$ such that $\min \rho(w_{evw}) \leq 0 < \min \rho(u)$ and thus $f \circ \rho(w^\sharp) < f(DR(w_{evw})^{-1} \min \rho(u)\mathbf{1}) < f \circ \rho(u)$ since $DR(w_{evw}) > 1$. All in all, there exists $w^\sharp \in \Pi^+$ such that $f \circ \rho(w^\sharp) < \max_{\mathbb{R}^n \backslash \{0\}} f \circ \rho$ *i.e.* $w^\sharp$ is not maximally ρ-presentative. Note that one cannot expect $\rho(w^\sharp) \prec \rho(w_{evw})$ as it contradicts Lemma 4.3. We recall (see (4.11)) that a maximally ρ-presentative w_f is such that $n^{-1}\langle \rho(w_f), \mathbf{1} \rangle \geq \min \rho(w^*)$. Then equipping $\mathbb{R}^n$ with the usual topology, let us consider $A := \{ w \in \mathbb{R}^n, w \succ 0 \}$, $F := \overline{A}$ its topological closure and the open set $O := \{ w \in \mathbb{R}^n \setminus \{0\}, n^{-1}\langle \rho(w), \mathbf{1} \rangle < \min \rho(w^*) \}$. So if $w \in O$ then w cannot be ρ-presentative. We verify $F \cap O \neq \emptyset$. Using twice Lemma 3.4 and then (3.1)

$$DR(w_{evw})\langle \rho(w^\sharp), w_{ew} \rangle = \varrho(w^\sharp, w_{evw}) \leq \min \rho(w_{evw}) \leq \langle \phi(w^*), \rho(w_{evw}) \rangle = \varrho(w_{evw}, w^*)/DR(w^*) \leq \min \rho(w^*)$$

hence $\langle \rho(w^\sharp), w_{ew} \rangle < \min \rho(w^*)$ which shows that $w^\sharp \in O \cap F$. However by [3, Chapitre 1, §1, Proposition 5], $O \cap F = O \cap \overline{A} \subset \overline{O \cap A}$ which proves that $w^\sharp \in \overline{O \cap A}$ and that $O_1 = O \cap A \neq \emptyset$. The fact that O_1 is an open set and that $O_1 \subset O$ proves our claim.

To prove that $\tilde{\mathcal{F}}$ is convex let us remark that $\tilde{\mathcal{F}} = \{ w \in \Pi^+,\ n^{-1}\langle \rho(w), \mathbf{1} \rangle \geq \min \rho(w^*) \}$ which is indeed convex by Proposition 2.4. To finish, let us now consider the equality case in the definition of $\tilde{\mathcal{F}}$. For any $w_f \in \mathcal{R}$, it can equivalently be written $\langle n^{-1}\mathbf{1}, \rho(w_f) \rangle = \min \rho(w^*)$. In this case (4.10) implies $f \circ \rho(w_f) = f \circ \rho(w^*)$ and as w_f is unique $w_f = w^*$ and thus $\langle n^{-1}\mathbf{1}, \rho(w^*) \rangle = \min \rho(w^*)$ which implies in turn $w_f = w^* = \bar{w}$. □

Before closing this section, let us give a few concluding remarks regarding the concept of maximally ρ-presentative portfolios. Firstly, in our analysis, the symmetry of the function f associated to such a portfolio is crucial. It is determined on financial ground since there is no reason to single out any asset before constructing a ρ-presentative portfolio. Now, without this assumption, it can be noted that any long-only portfolio θ would solve (4.1), as it would maximize the increasing and linear function $f = \langle \cdot, \phi(\theta) \rangle$.

Secondly, a parallel can be drawn between the mean-variance utility criterion used for portfolio construction [13] and the objective maximized in this section. Indeed, the function f being increasing by assumption, it will tend to favor portfolios with a higher average exposure. Also, as f is symmetric and concave, it is Schur concave (see [14]). Therefore, for portfolios having a given average exposure, those that have exposures that are "less spread out" (in the words of Marshall *et al.* [14]) will also be favored.

In a nutshell, one could view each f generating a maximally ρ-presentative portfolio as providing a particular trade-off between the average, the dispersion and possibly higher moments of the spectrum of a portfolio. To illustrate this idea, denoting respectively $\mathbb{E}(v)$ and $\mathbb{V}ar(v)$ the mean and variance of $v \in \mathbb{R}^n$, let us note that

$$f(\rho(w)) = \mathbb{E}(\rho(w)) - \frac{\lambda}{2}\mathbb{V}ar(\rho(w))$$

is an objective that is concave symmetric if $\lambda \geq 0$ (hence Schur concave) but increasing only for $\lambda < 1$. Remark that the rightmost term could also be modified to take into account interactions between exposures.

The case $\lambda \geq 1$ is excluded in the previous equation as the dominant term $-\mathbb{V}ar(x)$ is not increasing even though it is Schur concave. This mitigates the use of this latter assumption alone.

Example 4.8. We conclude this section with Figures 3 and 4 where we depict the sets of maximally ρ-presentative portfolios $\mathcal{R}$ that we got for respectively three and four assets whose covariance matrices are

$$\Sigma_1 = \begin{pmatrix} 1 & -0.4 & -0.8 \\ -0.4 & 1 & 0.7 \\ -0.8 & 0.7 & 1 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 1 & -0.05 & -0.7 & 0.2 \\ -0.05 & 1 & -0.3 & -0.6 \\ -0.7 & -0.3 & 1 & 0.5 \\ 0.2 & -0.6 & 0.5 & 1 \end{pmatrix}.$$

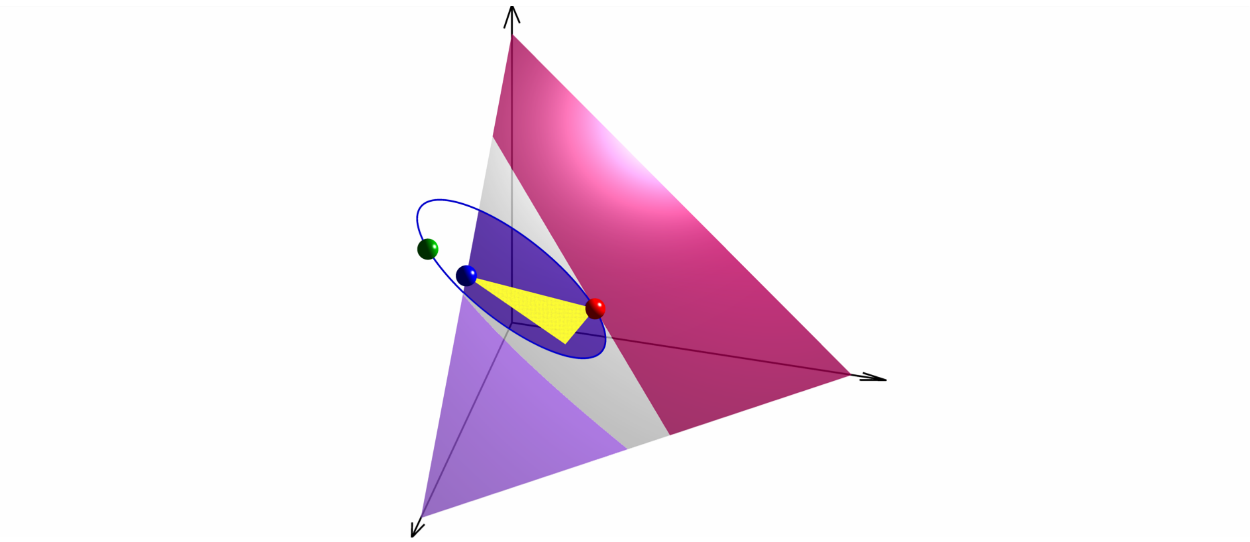

Figure 3: We represent the regular simplex Π^+ along with the set of maximally ρ-presentative portfolios $\mathcal{R}$ (in yellow), the sets $\mathcal{F}$ (in dark violet) and $\tilde{\mathcal{F}}$ (whose complement in Π^+ is indicated in light violet). From left to right the bullets depict the long-short MDP, the MDP and the EVW with the latter two being maximally ρ-presentative as we are going to see. Note that the long-short MDP and the EVW lie on the boundary of the ellipsoid that determines $\mathcal{F}$. The tangent hyperplane to $\mathcal{F}$ at w_{evw} separates Π^+ in two sets such that one of them (depicted in pink) does not contain any maximally ρ-presentative portfolio.

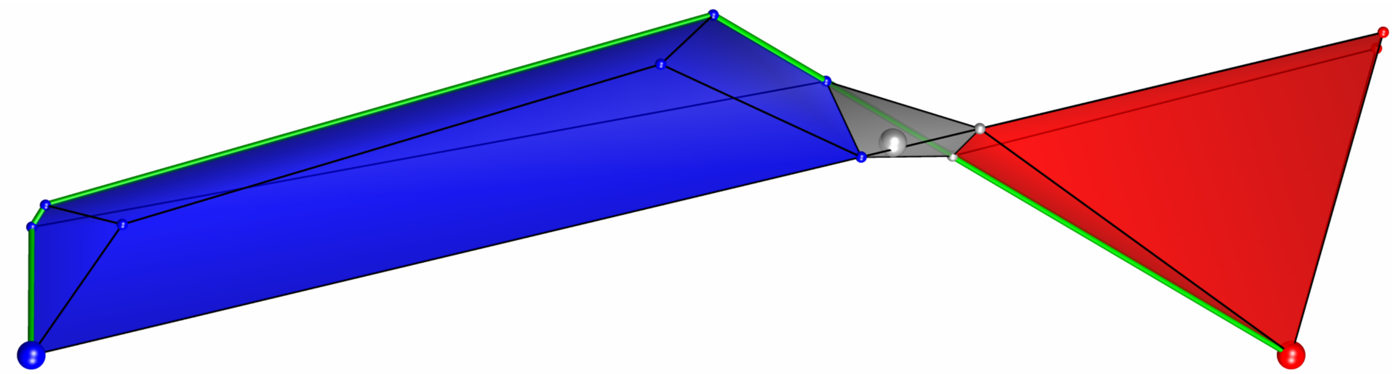

Figure 4: We represent the set of maximally ρ-presentative portfolios $\{(w_1, w_2, w_3) \in [0,1)^3 \ / \ (w_1, w_2, w_3, 1 - w_1 - w_2 - w_3) \in \mathcal{R}\}$ that is the union of three polytopes. From left to right the large bullets depict the MDP, ERC and EVW that are indeed maximally ρ-presentative as we are going to see. In Section 4.3.1, we shall also see that the constrained MDPs define a continuous path (depicted in green) within $\mathcal{R}$ that connects the EVW and the MDP.

4.2 Alternative Definitions of Well-Known Portfolios and Implications

As shown in Section 3, the MV, MDP and ERC are ρ-presentative. We investigate in this section whether these portfolios and the EVW are maximally ρ-presentative, yielding alternative definitions for these portfolios.

4.2.1 The Equal Volatility Weighted Portfolio

To improve the overall exposure of a portfolio one may maximize the average of its correlations to the assets:

Proposition 4.9. *The EVW is maximally ρ-presentative as it is the unlevered portfolio that maximizes its average correlation to all the assets amongst all non-zero long-short portfolios. Said otherwise,*

$$w_{evw} = \underset{w \in \Pi}{\operatorname{argmax}} \langle \rho(w), \mathbf{1} \rangle.$$

Proof. For $w \in \Pi$, $\langle \mathbf{1}, \rho(w) \rangle = \sigma_\Sigma(w)^{-1} \langle 1 \oslash \sigma, \Sigma w \rangle = \sigma_\Sigma(\mathbf{1} \oslash \sigma) \varrho(w, w_{evw})$ that is maximized by w_{evw} and any other such unlevered portfolio is perfectly correlated to it and thus identical by Proposition 1.1. □

The fact that the optimum over long-short portfolios is attained by a unique long-only portfolio can also be derived from Lemma 4.3 and Proposition 4.2 without exhibiting the solution. Moreover, let us recall that even though the EVW is maximally ρ-presentative it is not necessarily ρ-presentative, as was shown in Proposition 3.3. However, as expected, it is weakly ρ-presentative as $\langle \mathbf{1}, \rho(w_{evw}) \rangle = \sigma_\Sigma(\mathbf{1} \oslash \sigma) > 0$.

4.2.2 The Most Diversified Portfolio

We may wonder whether it is possible to build a portfolio that is both ρ-presentative and maximally ρ-presentative. For a positive answer, let us focus on portfolios that maximize their minimal exposure:

Proposition 4.10. *The MDP w^* is the unlevered portfolio that maximizes its minimal correlation to all assets amongst all long-short portfolios. Moreover, amongst long-short portfolios, the MDP is the unlevered portfolio that maximizes the minimum correlation to any long-only portfolio. Said otherwise,*

$$\underset{w \in \Pi^+}{\operatorname{argmax}} DR(w) = \underset{w \in \Pi}{\operatorname{argmax}} \min \rho(w) = \underset{w \in \Pi}{\operatorname{argmax}} \min_{\theta \in \Pi^+} \varrho(w, \theta). \tag{4.14}$$

In fact, $\forall (y, w) \in (\mathbb{R}^n \setminus \{0\}, \Pi^+)$, $\min \rho(y) \leq \min \rho(w^*) = \min_{\theta \in \Pi^+} \varrho(w^*, \theta) = DR(w^*)^{-1} \leq DR(w)^{-1}$.
In addition to being ρ-presentative, the MDP is also maximally ρ-presentative.

Proof. We start with the first claim of the proposition. Let $w \in \mathbb{R}^n \setminus \{0\}$ then given that $\phi(w^*) \in \Pi^+$,

$$\min \rho(w) \leq \langle \phi(w^*), \rho(w) \rangle = DR(w^*)^{-1} \varrho(w, w^*) = \min \rho(w^*) \varrho(w, w^*) \tag{4.15}$$

where we employed the last part of Lemma 3.4 and identity (3.1). Taking the supremum on Π on both ends proves that it is attained by w^* and any other such portfolio $y^* \in \Pi$ satisfies $\min \rho(w^*) = \min \rho(y^*) \leq \min \rho(w^*) \varrho(y^*, w^*)$ and is thus perfectly correlated to w^* as $\min \rho(w^*) > 0$ by (3.1) hence $y^* = w^*$ by Proposition 1.1. This finishes the proof of the first statement of the proposition.

It remains to prove the second identity in (4.14). As $\min_{\theta \in \Pi^+} \varrho(w, \theta) \leq \min \rho(w)$ with equality for $w = w^*$ by the identity in Lemma 3.4, one has $w^* \in \operatorname{argmax}_{w \in \Pi} \min_{\theta \in \Pi^+} \varrho(w, \theta)$. Now for any y^* in the rightmost set,

$$0 < \min \rho(w^*) = \min_{\theta \in \Pi^+} \varrho(w^*, \theta) = \min_{\theta \in \Pi^+} \varrho(y^*, \theta) \leq \min \rho(y^*)$$

which proves that y^* is ρ-presentative so by Lemma 3.4 the last inequality is an identity. Then taking $w = y^*$ in (4.15) and simplifying by $\min \rho(y^*) = \min \rho(w^*)$ on both ends, we obtain $\varrho(y^*, w^*) \geq 1$ hence $y^* = w^*$ by Proposition 1.1. This finishes the proof of the second statement of this proposition. □

This proposition proves that the MDP is the portfolio that maximizes its minimal exposure to all long only portfolios. As such, the *MDP maximizes its lowest exposure to all long-only factors*, defined as factors that are replicable by leveraged long-only portfolios of assets belonging to the universe.

Remark 4.11. In view of these results, one could think of constructing long-only portfolios that minimize their maximal exposure, in the spirit of a minimum variance approach. Formally, one may do so by solving

$$\min_{w\in\Pi^+} \max \rho(w).$$

This problem that we already encountered in Proposition 3.5 may admit many local minima and not necessarily a unique global solution. This makes this approach challenging when reaching the implementation phase in a financial setting. Furthermore, the set of optima of the min-max problem may not contain the solution of the max-min problem. One may verify numerically both of these remarks by considering three assets with $\Sigma = C$, $C_{1,2} = C_{1,3} \geq 0.7$, $C_{2,3} < 0.4$ and $C \succ 0$.

4.2.3 The Equal Risk Contribution Portfolio

Having considered some basic functions f, we pursue with the natural logarithm to prove that the ERC is maximally ρ-presentative:

Proposition 4.12. *The ERC is maximally ρ-presentative as*

$$w_{erc} = \operatorname*{argmax}_{w\in\Pi} \langle \ln(\rho(w)), \mathbf{1}\rangle$$

where the natural logarithm is taken entry-wise with the convention $\ln \equiv -\infty$ *on* $[-1, 0]$.
Furthermore,

$$DR(w_{evw}) \leq DR(w_{erc}) \leq DR(w^*) \quad \text{and} \quad \varrho(w_{erc}, w_{evw}) \geq \frac{DR(w_{evw})}{DR(w^*)}.$$

Proof. Consider $f : \mathbb{R}^n \to [-\infty, 0]$ defined by $x \mapsto \langle \ln(x), \mathbf{1}\rangle$ with the convention $\ln \equiv -\infty$ on $(-\infty, 0]$. As f admits infinite values its domain is different from $\mathbb{R}^n$ so we need to show that $\sup_\Pi f \circ \rho$ is indeed attained. As there exists a ρ-presentative portfolio $u \in \Pi^+$, there exists $\varepsilon > 0$ such that $\rho(u) > \varepsilon\mathbf{1}$ and thus $\langle \ln(\rho(u)), \mathbf{1}\rangle > n\ln(\varepsilon)$. So we can narrow our search to $\{w \in \Pi, \langle \ln(\rho(w)), \mathbf{1}\rangle \geq n\ln(\varepsilon)/2\}$ which is bounded and closed - by continuity of $w \mapsto \langle \ln(\rho(w)), \mathbf{1}\rangle$ - and thus a compact. This justifies that the sup is attained.

To deal only with finite values in the objective, we can add the non-binding constraint $\langle \ln(\rho(w)), \mathbf{1}\rangle \geq \ln(\varepsilon)$ in the maximization problem. However as any portfolio w that satisfies this constraint is such that $\prod_{i=1}^n \rho(w)_i \geq \varepsilon$ with $\rho(w)_i \in (0, 1]$, necessarily $\rho(w) \succeq \varepsilon\mathbf{1}$ which in turn implies that $\Sigma w \succeq \sigma_\Sigma(w)\varepsilon\sigma \succeq (\min_\Pi \sigma_\Sigma)\varepsilon\sigma$. The objective remains finite under this new constraint which is less restrictive and not binding either.

Moreover, the optimization is performed over Π and by 0-homogeneity of ρ, this corresponds to three exclusive cases: either $\langle \mathbf{1}, w\rangle = 1$, or $\langle \mathbf{1}, w\rangle = 0$, or $\langle \mathbf{1}, w\rangle = -1$. By Lemma 4.3, given a long-short portfolio there is always a long-only portfolio that improves the objective so we know that any solution is in Π^+, as a consequence we can discard the two latter non-binding constraints and keep only $\langle \mathbf{1}, w\rangle = 1$.

To sum up, we justified the following identity

$$\max_{w\in\Pi} \langle \ln(\rho(w)), \mathbf{1}\rangle = \max\left\{\langle \ln(\rho(w)), \mathbf{1}\rangle \;/\; \langle w, \mathbf{1}\rangle = 1,\ \Sigma w \succeq \alpha\sigma\right\}$$

where we set $\alpha = (\min_\Pi \sigma_\Sigma)\varepsilon$. The objective is finite and continuously differentiable over a set of linear constraints so we may apply the KKT theorem. However, as $\Sigma w \succeq \alpha\sigma$ is not binding, it will not appear in the KKT conditions. Now, denoting w the maximizer of the latter problem, the KKT condition reads $\nabla_w \left(w \mapsto \frac{n}{2}\ln(\sigma^2(w)) - \langle \ln(\Sigma w), \mathbf{1}\rangle\right)(w) = \mu\mathbf{1}$ with $\mu \in \mathbb{R}$. Computing the differential, we get

$$n\frac{\Sigma w}{\sigma^2(w)} = \Sigma(\mathbf{1} \oslash \Sigma w) + \mu\mathbf{1}$$

and then taking the inner product with w we observe that $\mu = 0$. Therefore, composing with Σ^{-1} we get $\Sigma w \odot w = n^{-1}\sigma^2(w)\mathbf{1}$ which, by [17, Corollary 1.2], is solved by a unique portfolio that is the ERC.

Finally, as $w_{erc} \in \mathcal{R}$, the remaining inequalities follow from Proposition 4.5. □

In this proposition, the inequalities between the Diversification Ratios of the EVW, ERC and MDP are the analog to those obtained with their volatilities in [12, Appendix A3]. Furthermore, as noted previously in Section 4.1, if $\mathbf{1}$ is an eigenvector of the correlation matrix the inequalities of the previous proposition imply that $w_{erc} = w_{evw} = w^*$ which is in this case the unique maximally ρ-presentative portfolio.

Now, let us remark that invoking Theorem 4.4, it is clear from the identity

$$\rho\left(w_{erc}\right) = n^{-1}\sigma\left(w_{erc}\right) \oslash \left(w_{erc} \odot \sigma\right) = n^{-1}DR^{-1}(w_{erc}) \oslash \phi(w_{erc})$$

that the ERC is indeed maximally ρ-presentative. However, we went through the effort of the previous analysis to not only give a direct proof but also exhibit a non-trivial objective that is a function of $\rho(w)$. In particular, this objective does not involve explicit long-only constraints as for usual formulations of this problem; which otherwise would lead to $2^n - 1$ non long-only solutions as shown in [17, Prop. 1.3]. In fact, this formulation suggests alternative ways of computing the ERC, that could complement the approach taken in [17].

4.2.4 The Minimum Variance Portfolio

In the same spirit as in the previous sections, we characterize MV portfolios using the spectrum $\rho(w)$:

Proposition 4.13. *One has*

$$\min_{w\in\Pi^+} \sigma(w) = \max_{w\in\mathbb{R}^n\setminus\{0\}} \min\left(\rho(w) \odot \sigma\right)$$

and the maximum is attained by a unique portfolio (up to leverage) that is the MV.
In fact, $\forall\,(y, w) \in (\mathbb{R}^n \setminus \{0\}, \Pi^+)$,

$$\min(\rho(y) \odot \sigma) \leq \min(\rho\left(w_{mv}\right) \odot \sigma) = \sigma\left(w_{mv}\right) \leq \sigma\left(w\right).$$

Furthermore, the MV is not necessarily maximally ρ-presentative.

Proof. Let $f : y \in \Pi \mapsto \min\left(\rho(y) \odot \sigma\right)$, then if $f(y) \geq \sigma(w_{mv})$, necessarily $\frac{\Sigma y}{\sigma(y)} \succeq \sigma(w_{mv})\mathbf{1}$ which implies $\rho(y, w_{mv}) = 1$ hence $y = w_{mv}$. This proves that $\{f \geq \sigma(w_{mv})\} \subset \{w_{mv}\}$. To check that the superlevel is not empty we remark that $f(w_{mv}) \geq \sigma(w_{mv})$. This follows from the KKT theorem applied to $\min_{\Pi^+}\sigma_\Sigma$ that shows that $\exists\lambda \succeq 0$ and $\Sigma w_{mv}/\sigma\left(w_{mv}\right) = \sigma\left(w_{mv}\right)\mathbf{1} + \lambda/\sigma(w_{mv})$ with $\lambda \odot w_{mv} = 0$ hence the claim.

Finally, to be maximally ρ-presentative, by Proposition-Definition 4.6, the MV needs to be weakly ρ-presentative and to satisfy the bound $\varrho(w_{mv}, w_{evw}) \geq \frac{DR(w_{evw})}{DR(w^*)}$. Consider a situation where all correlations are identical: then $w_{evw} = w^*$ and as a consequence we also have $w_{mv} = w^*$. Writing the KKT conditions satisfied by w^* and w_{mv} implies readily that $\mathbf{1}$ is an eigenvector of $CD_\sigma\Sigma^{-1} = CD_\sigma D_\sigma^{-1}C^{-1}D_\sigma^{-1} = D_\sigma^{-1}$ which leads to a contradiction if we consider assets that have different volatilities. ☐

4.3 On the Impact of Maximum Weight Constraints

In practice, asset managers may use maximum weight constraints when imposed by regulators or when using objective functions that are very sensitive to the estimation of their parameters (a common problem for long-short mean-variance maximization). To address this issue, robust covariance estimators are routinely used by asset managers with some popular choices involving shrinkage methods [11] or factor models [5].

The use of maximum weight constraints and robust covariance estimators can be closely related. Indeed, [9, Proposition 1] shows that, for the MV portfolio, imposing nonnegative and maximum weight constraints is equivalent to using a robust version of the original covariance matrix. This matrix is robust in the sense that extreme covariances are the most likely to be "shrunk" towards more reasonable values. A limitation of the method is that the modified matrix depends on Lagrange multipliers that are known only after the MV optimization or determined through a numerically demanding maximization of a likelihood function over a set of matrices (cf. [9, Proposition 2]).

Another route proposed in this paper is to identify *a priori* an essentially unconstrained optimization problem whose objective depends *explicitly* on the maximum weight constraint, and is equivalent to the original constrained problem. This new problem gives as a result a clear understanding of the maximum weight constraint on the original objective function. The constrained portfolios we consider here have a *volatility-adjusted maximum weight constraint*, that is they belong to

$$\Pi^+_{\sigma,r} := \left\{w \in \Pi^+ \;/\; \forall i \in \{1, \cdots, n\},\; \frac{w_i\sigma_i}{\langle w, \sigma\rangle} \leq \frac{1}{r}\right\}$$

for some real r. In particular, portfolios with *maximum weight constraint* $1/r$ belong to $\Pi^+_{\mathbf{1},r}$.

We first present the unconstrained optimization problems that are equivalent to the original problems solved by the MDP and MV respectively, and conclude this subsection by discussing the implications of these two results.

4.3.1 An Alternative Definition of the Constrained Most Diversified Portfolio

We consider in this section an aggregation of the correlation spectrum that generalizes those proposed in Propositions 4.9 and 4.10 for the EVW and the MDP.

Definition 4.14. For $r \in \{1, \cdots, n\}$, the *rank-r ρ-presentativity measure* of $w \in \mathbb{R}^n \setminus \{0\}$, denoted $RM_r(w)$, is the average of the r smallest correlations to the assets. Considering the reordering $(\rho(w))_{(i)} \leq (\rho(w))_{(i+1)}$,

$$RM_r(w) := \frac{1}{r}\sum_{i=1}^{r} (\rho(w))_{(i)}.$$

Using the lingo of Section 4.1, $RM_r(w) = \langle \rho(w)^{\downarrow}, r^{-1}\mathbf{1}_r \rangle$ where $\mathbf{1}_r$ is the vector whose r first coordinates are equal to one and zero elsewhere. We could also consider real-valued $r \in [1, n]$ thanks to the identity $\langle \rho(w)^{\downarrow}, r^{-1}\mathbf{1}_r \rangle = \min_{\theta \in \Pi^+_{\mathbf{1},r}} \langle \rho(y), \theta \rangle$.

The average of the r smallest elements of a vector, is concave increasing and symmetric. We show that the constrained MDP w^*_r, that maximizes DR over $\Pi^+_{\sigma,r}$, is also the portfolio that maximizes RM_r. This therefore implies that it is maximally ρ-presentative and that it can be obtained by an *unconstrained optimization* of an objective that incorporates the long-only and volatility-adjusted maximum constraints:

Proposition 4.15. *The constrained MDP w^*_r is maximally ρ-presentative as it is the unlevered portfolio that maximizes the rank-r ρ-presentativity measure RM_r over non-zero long-short portfolios. Said otherwise*

$$\underset{w \in \Pi^+_{\sigma,r}}{\operatorname{argmax}}\, DR(w) = \underset{w \in \Pi}{\operatorname{argmax}}\, RM_r(w). \tag{4.16}$$

In fact,

$$\forall (y, w_r) \in \left(\mathbb{R}^n \setminus \{0\}, \Pi^+_{\sigma,r}\right), \ RM_r(y) \leq RM_r(w^*_r) = DR(w^*_r)^{-1} \leq DR(w_r)^{-1}. \tag{4.17}$$

To prove this proposition we shall use properties of the constrained MDP that relate both DR and RM_r:

Proposition 4.16. *w^*_r exists, is unique and $DR(w^*_r)RM_r(w^*_r) = 1$. In addition, $\forall (y, w_r) \in \left(\mathbb{R}^n \setminus \{0\}, \Pi^+_{\sigma,r}\right)$,*

$$RM_r(y) \leq \varrho(y, w^*_r)\, RM_r(w^*_r), \tag{4.18}$$

$$DR(w_r) \leq \varrho(w_r, w^*_r)\, DR(w^*_r). \tag{4.19}$$

Having this proposition at our disposal, we are ready to prove the proposition:

Proof of Proposition 4.15. The existence of w^*_r follows from Proposition 4.16. Taking the supremum on both sides of (4.18) shows that w^*_r attains it so w^*_r is maximally ρ-presentative, and all portfolios achieving the supremum are perfectly correlated to it. By Proposition 1.1, the MDP is the unique unlevered portfolio that maximizes RM_r. Remaining results follow directly from Proposition 4.16. □

Proof of Proposition 4.16. The function ϕ introduced before Proposition 2.5 is a bijection from $\Pi^+_{\sigma,r} \to \Pi^+_{\mathbf{1},r}$, and $DR(w) = \sigma_C(\phi(w))^{-1}$. Now as $\Pi^+_{\sigma,r}$ and $\Pi^+_{\mathbf{1},r}$ are compact, and DR and σ_C are continuous on these sets, they reach their extrema and one can write $\phi\left(\operatorname{argmax}_{\Pi^+_{\sigma,r}} DR\right) = \operatorname{argmin}_{\Pi^+_{\mathbf{1},r}} \sigma_C$. Taking x^* in the rightmost set, by Proposition 2.5, we just need to establish $\frac{1}{r}\sum_{i=1}^{r}(\rho_C(x^*))_{(i)} = \sigma_C(x^*)$ to prove the first claim. To do so, the idea is to find the average of the r smallest entries of Cx^* by applying the KKT theorem to $\min_{\Pi^+_{\mathbf{1},r}} \sigma_C$, which, as $C \succeq 0$, implies that there exist $\lambda \succeq 0,\ \mu \succeq 0$ such that $x^* \in \Pi^+_{\mathbf{1},r}$ verifies the KKT conditions

$$Cx^* = s\mathbf{1} + \lambda - \mu,\ \lambda \odot x^* = 0 \text{ and } \mu \odot \left(r^{-1}\mathbf{1} - x^*\right) = 0.$$

On the one hand, these conditions imply that $\sigma_C^2(x^*) = s - \langle \mu, x^* \rangle$ and $\langle \mu, x^* \rangle = r^{-1}\langle \mathbf{1}, \mu \rangle$, so $s - r^{-1}\langle \mathbf{1}, \mu \rangle = \sigma_C^2(x^*)$. On the other hand, the two last KKT conditions yield three mutually exclusive cases:

$$\begin{cases} x_i^* = 0 & \Rightarrow (\lambda_i \geq 0 \quad \text{and} \quad \mu_i = 0) \Rightarrow \lambda_i - \mu_i \geq 0 \quad \text{(Case 1)}, \\ 0 < x_i^* < r^{-1} & \Rightarrow (\lambda_i = 0 \quad \text{and} \quad \mu_i = 0) \Rightarrow \lambda_i - \mu_i = 0 \quad \text{(Case 2)}, \\ x_i^* = r^{-1} & \Rightarrow (\lambda_i = 0 \quad \text{and} \quad \mu_i \geq 0) \Rightarrow \lambda_i - \mu_i \leq 0 \quad \text{(Case 3)}. \end{cases}$$

As $x^* \in \Pi_{\mathbf{1},r}^+$, $\#\{x_i^* > 0\} \geq r$, so the sum of the r smallest entries of $\lambda - \mu$ is obtained through summation of all the elements of $-\mu$ only (Cases 2 and 3). Therefore,

$$\frac{\sigma_C(x^*)}{r}\sum_{i=1}^{r}(\rho_C(x^*))_{(i)} = \frac{1}{r}\sum_{i=1}^{r}(Cx^*)_{(i)} = \frac{1}{r}\sum_{i=1}^{r}(s\mathbf{1} + \lambda - \mu)_{(i)} = s + \frac{1}{r}\sum_{i=1}^{r}(-\mu)_{(i)} = s - r^{-1}\langle \mathbf{1}, \mu \rangle = \sigma_C^2(x^*)$$

which proves that $DR(w_r^*)RM_r(w_r^*) = 1$. To finish, as $C \succ 0$, uniqueness of w_r^* comes from that of x^*.

Now let us turn to the proof of (4.18) and (4.19): by the last claim of Lemma 3.4

$$\forall (y, w_r) \in \left(\mathbb{R}^n \setminus \{0\}, \Pi_{\sigma,r}^+\right), \varrho(y, w_r) = DR(w_r)\langle \phi(w_r), \rho(y) \rangle \geq DR(w_r) \min_{\theta \in \Pi_{\mathbf{1},r}^+} \langle \theta, \rho(y) \rangle = DR(w_r)\, RM_r(y).$$

Using $RM_r(w_r^*)\, DR(w_r^*) = 1$, the two inequalities follow if we take in turn $w_r = w_r^*$ and then $y = w_r^*$. ☐

On the practical side, this proposition provides the "duality gap" (4.17) which makes it possible to assess the optimality of a long-only portfolio in terms of DR without computing the MDP. Indeed for any $w \in \Pi_{\sigma,r}^+$,

$$0 \leq DR(w_r)^{-1} - DR(w_r^*)^{-1} \leq DR(w_r)^{-1} - RM_r(w)$$

where on the right-hand side we do not use w^*. This can be useful in an algorithm as a stopping criterion.

Remark 4.17. *(i)* We may conclude that w_r^* is maximally ρ-presentative by invoking Theorem 4.4 once the identity $DR(w_r^*)RM_r(w_r^*) = 1$ is established.

(ii) For long-only portfolios neither of the two inequalities (4.18) and (4.19) is superior to the other. Indeed, consider three assets with $\Sigma = C$, $C_{1,2} = C_{1,3} = 0.7$, $C_{2,3} = 0.3$ and $r = 1$. Then the sign of

$$\frac{DR(w)}{DR(w_r^*)} - \frac{RM_r(w)}{RM_r(w_r^*)} = \frac{\sigma(w^*)}{\sigma(w)} - \frac{\min(\rho(w))}{\sigma(w^*)}$$

flips when picking $w \in \{e_1, e_2\}$. However (4.18) is more general as it holds for long-short portfolios.

(iii) Moving to another topic, assuming that Σ is positive semi-definite is enough to derive the KKT conditions in the proof of Proposition 4.16. Under this weaker hypothesis, (4.16) can be established along the same lines as an identity between sets. The definiteness comes into play to prove that the MDP is unique and that it is the unique portfolio that maximizes RM_r by Proposition 1.1. Dropping the definiteness of Σ, we still know that all portfolios in the maximizing sets are perfectly correlated. One has to be careful and select $w \in \Pi \setminus \mathrm{Ker}(\Sigma)$ to avoid dividing by zero in the definition of $\rho(w)$. From the beginning, one could have actually balanced the definition of Π and the class of matrices that are allowed by picking them in $\{\Sigma \succeq 0 \;/\; \sigma_\Sigma > 0 \text{ on } \Pi\}$.

4.3.2 An Alternative Definition of the Constrained Minimum Variance

Before concluding this section, we state a generic result that yields Proposition 4.15 in the special case $\Sigma = C$ and that is obtained along the same lines:

Theorem 4.18. *The minimization of a positive definite quadratic form over the simplex subject to a uniform maximum constraint can be expressed as an unconstrained optimization as follows:*

$$\min_{w \in \Pi_{\mathbf{1},r}^+} \sigma(w) = \max_{w \in \mathbb{R}^n \setminus \{0\}} \frac{1}{r}\sum_{i=1}^{r}(\rho(w) \odot \sigma)_{(i)},$$

where the maximum is attained by a unique unlevered portfolio that is the constrained MV $w_{mv,r}$ (the EW if $r = n$). In fact,

$$\forall (y, w) \in \left(\mathbb{R}^n \setminus \{0\}, \Pi_{\mathbf{1},r}^+\right), \; \frac{1}{r}\sum_{i=1}^{r}(\rho(y) \odot \sigma)_{(i)} \leq \frac{1}{r}\sum_{i=1}^{r}(\rho(w_{mv,r}) \odot \sigma)_{(i)} = \sigma(w_{mv,r}) \leq \sigma(w).$$

This gap can be used to assess the optimality of a portfolio without computing the constrained MV.

Furthermore, the constrained MV $w_{mv,r}$ (the EW if $r = n$) is not necessarily maximally ρ-presentative.

Proof. Conducting an analysis similar to the previous section, one obtains the first two assertions. It remains to prove that the constrained MV is not necessarily maximally ρ-presentative. The case $r = 1$ that corresponds to the unconstrained MV was handled in the proof of Proposition 4.13. To be maximally ρ-presentative, by Proposition-Definition 4.6, the constrained MV needs to be weakly ρ-presentative and to satisfy the bound $\varrho(w_{mv,r}, w_{evw}) \geq \frac{DR(w_{evw})}{DR(w^*)}$. Consider a situation where all correlations are identical. Then $w_{evw} = w^*$ hence $w_{mv,r} = w_{evw}$. Now, if $r > 1$, then by sending the volatility of a single asset to zero, its weight in the EVW can be made as close to one as one wishes. In this situation, the constrained MV whose weights are bounded by $1/r$ is in general different from the EVW. Taking $r = n$ proves that the EW is not maximally ρ-presentative. □

This result shows that the constrained MV or the EW maximize an aggregated exposure, where individual exposures are given by $\rho(w) \odot \sigma$. These are usually called *marginal risk contributions* (see [16]). As such, using $\rho(w) \odot \sigma$ as an alternative measure of exposure would lead to a new framework, where these portfolios would indeed be maximally exposed. Conducting an analysis similar to the proof of Theorem 4.4, we can prove that the set of maximally exposed portfolios given this measure of exposures is exactly

$$\mathcal{R}_\sigma := \{w \in \Pi^+,\ \sigma(w)^2 = \langle w^\uparrow, (\Sigma w)^\downarrow \rangle\}$$

which is small in the sense of Theorem 4.4 and for any $w \in \mathcal{R}_\sigma$, $\varrho(w, w_{ew}) \geq \sigma(w)/\sigma(w_{ew})$. In a similar way, one can carry the results of Theorem 4.4 to the set $\mathcal{R}_\mu$ that is associated to a general weighted measure $\rho(w) \odot \mu$ with $\mu \succ 0$. In addition to our discussion before Example 4.8, this offers another alternative to the celebrated approach of Markowitz and is left for further research. Getting to the main subject, the fact that the constrained MV is not maximally ρ-presentative can also be understood in the context of [7] where it shown that the MV and EW are *not leverage invariant*, as opposed to the ERC, EVW and MDP.

4.3.3 Implications of these Alternative Definitions

The results obtained in Sections 4.3.1 and 4.3.2 allow to identify *a priori* how the objectives maximized by the MDP and MV are modified by the addition of maximum weight constraints, that are volatility-adjusted for the MDP. Consider for example the case of the MV portfolio in a universe of 500 assets. Theorem 4.18 shows that minimizing the volatility of a long-only portfolio is equivalent to maximizing the minimal marginal risk contribution of a long-short portfolio with weights summing to one (cf. Lemma 4.3 and the proof of Proposition 4.12). Moreover, if a maximum weight constraint of 2% is added, the problem becomes equivalent to the maximization of the average of the lowest 50 marginal risk contributions of such a long-short portfolio.

A related result is provided in [9, Proposition 1], whereby the problem of minimizing the volatility of a long-short portfolio whose weights sum to one is studied. The authors show that adding minimum and maximum weights to this problem is equivalent to solving the original problem using a modified covariance matrix that is clearly identified. Nevertheless, its analytical form is *a priori* unknown as it depends on the Lagrange multipliers associated to the added constraints. However, the authors provide an interpretation of this modified matrix, and show that the adjustment brought to the original matrix "may reduce [its] estimation error". In the remaining of [9], an empirical study is conducted that indeed confirms these claims.

A connection between the results provided in [9] and Proposition 4.15 and Theorem 4.18 can easily be made in a context where the covariance matrix of the assets needs to be estimated. In this case, the correlation spectrum $\rho(w)$ and the marginal risk contributions $\min(\rho(w) \odot \sigma)$ are subject to estimation errors. Coming back to our MV example, this means that adding a 2% maximum weight constraint is equivalent to maximizing an objective that now averages 50 estimated variables. This arguably may contribute to "reduce [its] estimation error". This reduction can of course be expected to come at a cost and introduce a bias. However, as so far, we have assumed that the covariance matrix is given, any further statistical or empirical analysis is beyond the scope of this paper and is left for future research.

4.4 An Alternative Framework For Constructing Portfolios

So far we have shown that many well-known - possibly constrained - investment strategies maximize their overall exposure to the assets, as measured by some real-valued f. This in fact provides a unifying framework, whereby all strategies maximize an unconstrained objective that is a function of the spectrum $\rho(w)$.

We summarize most of these results in Table 1. Given an investment strategy that is indicated in the first column, the second column provides the *primal objective* that is maximized by the corresponding portfolio, while the third column contains its well-known *dual definition.* The use of the primal-dual terminology is justified in Section 5.2. The following columns then indicate whether the considered portfolio is always long-only, ρ-presentative or maximally ρ-presentative. Key remarks and references are indicated in the last column.

We have also included in the table three portfolios that are obtained using functions f that do not satisfy at least one of the three assumptions of Definition 4.1. The first such portfolio is the first eigenvector of the covariance matrix Σ. It is obtained with $f(w) = \|\rho(w) \odot \sigma\|_2$. Indeed,

$$\max_{w\in\mathbb{R}^n\setminus\{0\}} \|\rho(w) \odot \sigma\|_2^2 = \max_{w\in\mathbb{R}^n\setminus\{0\}} \frac{\langle \Sigma w, \Sigma w\rangle}{\langle \Sigma w, w\rangle} = \max_{w\in\mathbb{R}^n\setminus\{0\}} \frac{\|w\|_2^2}{\langle \Sigma^{-1} w, w\rangle}$$

which is solved by the eigenvector of Σ associated to its largest eigenvalue. This function does not satisfy any of our assumptions and the resulting portfolio maximizes an aggregation of its *absolute exposures* rather than its exposures. Secondly, we called "Assets" the portfolios reduced to single assets that are obtained by maximizing $f(x) = \sum_{i=1}^n x_i^p$ when we specialize $\Sigma = I$ and consider $p \in (2, +\infty]$. Indeed, for any such p and any $x \in \mathbb{R}^n \setminus \{0\}$, $f(x) \le \|x\|_p^p \le \|x\|_2^p$ with equality between these terms occurring only at elements of the canonical basis. This shows that multiple solutions may be obtained when using a function f that is not concave. Finally, "generic LO", is a generic portfolio that is different from the EVW portfolio and is obtained with a function f that depends on the portfolio weights and is thus not symmetric.

Investment Strategy Name	Primal approach: Portfolios maximize $f \circ \rho(w) =$	Dual approach: Weights proportional to	LO	ρ-pr	max ρ-pr	Remarks and References
EW	$\langle \rho(w) \odot \sigma, \mathbf{1}\rangle$	$\mathbf{1}$	×			cf. Prop. 3.3 and Thm. 4.18
EVW	$\langle \rho(w), \mathbf{1}\rangle$	$\mathbf{1} \oslash \sigma$	×		×	cf. Propositions 3.3 and 4.9
generic LO	$\langle \rho(w), \phi(\theta)\rangle$	$\theta \in \Pi^+ \setminus \{w_{evw}\}$	×			f not sym, cf. end of Section 4.1
ERC	$\langle \ln(\rho(w)), \mathbf{1}\rangle$	$w_i(\Sigma w)_i = \frac{\sigma^2(w)}{n}$	×	×	×	cf. Propositions 3.3 and 4.12
MV	$\min \rho(w) \odot \sigma$	$\operatorname{argmin}_{\Pi^+} \sigma_\Sigma$	×	×		f not sym, cf. Prop 3.3 and 4.13
MDP	$\min \rho(w)$	$\operatorname{argmax}_{\Pi^+} DR$	×	×	×	cf. Propositions 3.3 and 4.10
constr MV	$\sum_{i=1}^r (\rho(w) \odot \sigma)_{(i)}$	$\operatorname{argmin}_{\Pi_{1,r}^+} \sigma_\Sigma$	×			f not sym, cf. Thm. 4.18
constr MDP	$\sum_{i=1}^r (\rho(w))_{(i)}$	$\operatorname{argmax}_{\Pi_{\sigma,r}^+} DR$	×		×	cf. Proposition 4.15
LS MDP	$-\mathbb{V}ar(\rho(w))$	$\pm\Sigma^{-1}\sigma$		×		Long-short and ρ-pr, cf. Prop 5.3
Assets	$\langle \rho(w)^p, \mathbf{1}\rangle,\ p > 2$	any e_i if $\Sigma = I$	×			f not concave, several maximizers
1st eigv of Σ	$\|\rho(w) \odot \sigma\|_2$	$\operatorname{argmax}_w \frac{\sigma(w)}{\|w\|_2}$				f not concave, 1st PCA factor of Σ
max ρ-pr θ	$\langle \rho(w)^{\downarrow}, \phi(\theta)^{\uparrow}\rangle$	$\theta \in \mathcal{R}$	×		×	cf. Theorem 4.15
mean-var ρ	$\mathbb{E}(\rho(w)) - \frac{\lambda}{2}\mathbb{V}ar(\rho(w))$		×		×	$\lambda \in [0, 1)$, cf. end of Section 4.1
$w^\sharp$	$-\langle \rho(w), \mathbf{1}\rangle - \delta_{\Pi^+}(w)$		×			never max ρ-pr, proof of Prop 4.7
min max ρ	$-\min \rho(w) - \delta_{\Pi^+}(w)$		×			not unique, Prop 3.5 and Rmk 4.11

Table 1: An Alternative Framework For Constructing Portfolios. We used the following abbreviations: constr: *constrained*, eigv: *eigenvector*, LO: *long-only*, LS: *long-short*, ρ-pr: *ρ-presentative*, max ρ-pr: *maximally ρ-presentative*, sym: *symmetric*. Finally, δ_{Π^+} denotes the function that vanishes on Π^+ and that is $+\infty$ elsewhere.

As seen in Table 1, the primal objectives use basic functions and we could think of using convex combinations of these objectives, in order to create a composite objective. This would allow to not only retrieve all the above portfolios using a single objective, but all the intermediary portfolios in the spirit of [10]. Nonetheless these portfolios are not necessarily maximally ρ-presentative. We recall that in Figure 4, we have depicted a path connecting the EVW to the MDP that resides entirely in the set of maximally ρ-presentative portfolios.

All the primal objectives shown in the above table are functions of $\rho(w)$. The correlation of a portfolio to all the assets of its investment universe is easily computed and one does not need to know the weights. This may prove useful in order to assess in an alternative manner, whether a given fund is close to realize its Primal objective, without knowing the holdings of the fund. We are going to pursue in this direction in Section 5.3.

5 Applications

5.1 The Core Properties of the Constrained MDP

This section is dedicated to a theoretical application of Proposition 4.15 that also uses some elements of the proof of Proposition 4.16. To be precise, we state two equivalent definitions of the constrained MDP - as defined in Section 4.3 - that extend to the constrained case the *first and second core properties* of [7].

Proposition 5.1 (First Core Property)**.** *The MDP w_r^* with volatility-adjusted maximum weight $1/r$,*

(i) is more or equally correlated to the assets it does not hold than to those it holds,
(ii) is more or equally correlated to the assets that do not saturate the max constraint within those it holds,
(iii) has an identical correlation to the assets that do not saturate the constraint within the assets it holds.

Conversely, any portfolio in $\Pi_{\sigma,r}^+$ that satisfies (i), (ii) and (iii) is necessarily the constrained MDP w_r^*.

Proof. Statement (i) reads $(w_r^*)_i = 0$ and $(w_r^*)_j > 0 \implies (\rho(w_r^*))_i \geq (\rho(w_r^*))_j$. It is enough to prove the results for $x^* = \phi^{-1}(w_r^*)$. Employing the KKT theorem as in the proof of Proposition 4.16 and using the same notations, $(Cx^*)_i = s + (\lambda_i - \mu_i) \geq s \geq s + (\lambda_j - \mu_j) = (Cx^*)_j$. In a similar way, we get (ii). Claim (iii) follows readily from (Case 2) in the same proof, that is, $\lambda_j = \mu_j = \lambda_i = \mu_i = 0$, hence $(Cx^*)_i = (Cx^*)_j$. Conversely, assume that $w = \phi(x)$ satisfies all claims that then imply that

$$0 = x_i < x_j \leq x_k = r^{-1} \implies (Cx)_i \geq (Cx)_j \geq (Cx)_k.$$

If m is the number of saturated stocks, $m \leq r$ since $\langle x, \mathbf{1} \rangle = 1$. Denoting I (resp. J) the indices of the stocks that saturate (resp. do not saturate) the constraint, then

$$\sigma_C(x)^2 = \sum_{i \in I} (Cx)_i x_i + \sum_{j \in J} (Cx)_j x_j = \frac{1}{r} \sum_{i=1}^{m} (Cx)_{(i)} + \sum_{j \in J} (Cx)_j x_j$$

by *(i)* and *(ii)*. Now given that by *(iii)*, $\exists \nu \in \mathbb{R} / \forall j \in J, (Cx)_j = \nu$, it follows

$$\sigma_C(x)^2 = \frac{1}{r} \sum_{i=1}^{m} (Cx)_{(i)} + \nu \sum_{j \in J} x_j = \frac{1}{r} \sum_{i=1}^{m} (Cx)_{(i)} + \nu \left(1 - \sum_{i \in I} x_i \right) = \frac{1}{r} \sum_{i=1}^{m} (Cx)_{(i)} + \nu \left(1 - \frac{m}{r} \right),$$

where in the two last identities we used the fact that $\langle x, \mathbf{1} \rangle = 1$ and the definition of I. However,

$$\frac{1}{r} \sum_{i=1}^{r} (Cx)_{(i)} = \frac{1}{r} \sum_{i=1}^{m} (Cx)_{(i)} + \frac{1}{r} \sum_{j=m+1}^{r} (Cx)_{(j)} = \frac{1}{r} \sum_{i=1}^{m} (Cx)_{(i)} + \frac{\nu}{r} (r - m)$$

by *(ii)* and *(iii)*. Thus, $\frac{1}{r} \sum_{i=1}^{r} (Cx)_{(i)} = \sigma_C(x)^2$ which concludes the proof by Proposition 4.15. ☐

Proposition 5.2 (Second Core Property)**.** *The following statements are equivalent:*

(i) w_r^* is the MDP with volatility-adjusted maximum weight constraint $1/r$,
(ii) $w_r^* \in \Pi_{\sigma,r}^+$ is such that for any $w_r \in \Pi_{\sigma,r}^+$, $DR(w_r) \leq \varrho(w_r, w_r^*) DR(w_r^*)$.

Proof. *(ii)* $\implies$ *(i)* as $\forall w_r \in \Pi_{\sigma,r}^+$, $DR(w_r) \leq \varrho(w_r, w_r^*) DR(w_r^*) \leq DR(w_r^*)$, *i.e.* w_r^* has the highest DR that can be achieved over the set of constraints $\Pi_{\sigma,r}^+$. *(i)* $\implies$ *(ii)* is simply inequality (4.19). ☐

5.2 A Not-So-Typical Saddle-Point Problem

In Proposition 4.10, we introduced the problem

$$\max_{w \in \mathbb{R}^n \setminus \{0\}} \min_{\theta \in \Pi^+} \varrho(w, \theta),$$

which may remind us of a minimax matrix game problem or *saddle-point problem* but it is different in nature as $(w, \theta) \mapsto \varrho(w, \theta)$ is not (even quasi) concave-convex. Let us pick an example with three assets with covariance

$$\Sigma = \begin{pmatrix} 1.0 & -0.3 & -0.4 \\ -0.3 & 1.0 & -0.5 \\ -0.4 & -0.5 & 1.0 \end{pmatrix}.$$

In this case the MDP and MV are both equal to $w^* \approx [0.31, 0.32, 0.36]$. This example is an opportunity to show the different nature of the objective functions in Proposition 4.10 where we actually proved

$$\max_{w \in \mathbb{R}^n \setminus \{0\}} \min \rho(w) = \max_{w \in \mathbb{R}^n \setminus \{0\}} \min_{\theta \in \Pi^+} \varrho(w, \theta) = \min_{w \in \Pi^+} \sigma(w).$$

These identities may remind us a primal-dual framework with the *primal* and *dual* problems on both ends.

One can argue as in the proof of Proposition 4.12, to reduce the search set to those long-short w that sum to one. Therefore, to illustrate these problems, we can depict in Figure 5 the levels lines of the objective functions $(x, y) \mapsto \sigma(x, y, 1 - x - y)$ and $(x, y) \mapsto \min \rho(x, y, 1 - x - y)$. As they are significantly different we also draw the level lines of $(x, y) \mapsto \min_{\theta \in \Pi^+} \varrho((x, y, 1 - x - y), \theta)$. This latter chart allows us to better understand the second identity of (4.14) in Proposition 4.10 which is implied by Lemma 3.4 and to shed some light on the remark that follows this lemma (see the caption of Figure 5 for the details). To further illustrate the duality suggested by the inequalities (4.17) we depict the graphs of the three functions we just mentioned.

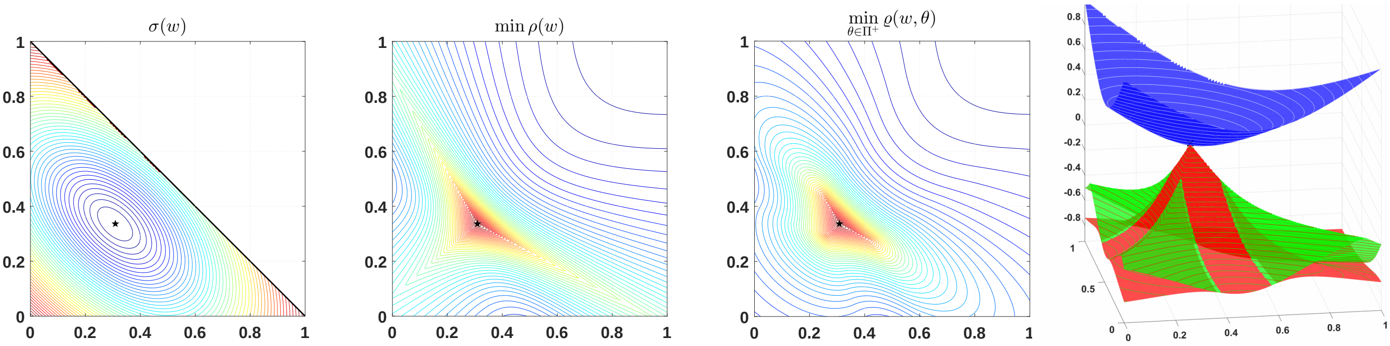

Figure 5: As indicated by the level lines, the MV/MDP depicted by a star minimizes $w \mapsto \sigma(w)$ while maximizing both the non-smooth $w \mapsto \min \rho(w)$ and $w \mapsto \min_{\theta \in \Pi^+} \varrho(w, \theta)$ as proven in Proposition 4.10. The non-convex superlevels indicate that the latter functions are not quasi-concave. From the two charts in the middle, it is clear that there are $w \in \Pi^+$ with $\min_{\theta \in \Pi^+} \varrho(w, \theta) < \min \rho(w)$. By Lemma 3.4, these portfolios are such that $\rho(w) \not\geq 0$. To the right, we plot at the top the graph of $w \mapsto \sigma(w)$ and below the graphs of $w \mapsto \min(\rho(w))$ and $w \mapsto \min_{\theta \in \Pi^+} \varrho(w, \theta)$ which are not smooth. We can observe that these three graphs intersect at a single point that is the MV/MDP.

5.3 Realized max ρ-presentativity and Realized Diversification

Let us recall that Proposition 4.5 asserts that any maximally ρ-presentative w satisfies the necessary condition

$$DR(w) \geq DR(w_{evw}) / \varrho(w, w_{evw}).$$

In this section, we show how this bound could be used to identify funds that qualify for being maximally ρ-presentative without knowing their composition. Indeed on the right hand side $\varrho(w, w_{evw})$ can be measured by simply computing the correlation between the time series of w and that of w_{evw}. The latter is computed thanks to the series of the assets of the universe. Lastly, using time series only, the realized DR of a portfolio with unknown composition can be also measured thanks to the following result already proved in (4.13):

Proposition 5.3. *Denoting $\bar{w} = \Sigma^{-1}\sigma / \|\Sigma^{-1}\sigma\|_1$ the portfolio that maximizes the DR over Π, for any $w \in \Pi$,*

$$DR(w) = DR(\bar{w}) \varrho(\bar{w}, w).$$

Thus the long-only MDP is the portfolio that is most correlated to the long-short MDP amongst all long-only portfolios. In this sense, the long-only MDP is the *projection* of the long-short MDP over long-only portfolios. Note that, using this identity, one can reformulate (4.19) in a way that may remind us a triangle inequality:

$$\forall w_r \in \Pi^+_{\sigma,r}, \;\; \varrho(w_r, \bar{w}) \leq \varrho(w_r, w_r^*) \, \varrho(w_r^*, \bar{w}).$$

Here, we used $\varrho(w_r^*, \bar{w}) \geq 0$ which follows from Proposition 5.3.

Let us get back to our idea, and perform a numerical experiment where we take as a universe 464 stocks of the MSCI USA (having discarded those that did not trade at least 90% of the days over 01/2013 to 03/2017). Using the Bloomberg Fund Screening module, we similarly considered daily time-series for funds that satisfy:

```
Market Status: Active                        Currency: USD                    Fund Primary Share Class: Yes
Fund Asset Class Focus: Equity               Fund Pricing Frequency: Daily    First Date: <= 1/1/2013
Fund Geographical Focus: International      Fund Strategy: Blend             Fund Total Assets (mil): >100M
```

We discarded 71 funds that had obvious price synchronization issues, ending up with 2278 funds for a total of \$7500bn *i.e.* about half the total net assets invested in the USA in Q1/2016. In Figure 6, we depict the realized $DR(w)$ of these funds as a function of $DR(w_{evw})/\varrho(w, w_{evw})$ and indicate the identity function using dashed line. A live fund depicted by a red star satisfies the necessary condition as it lies above the dashed line, as do the forward looking constrained MDPs that are indeed maximally ρ-presentative by Proposition 4.15.

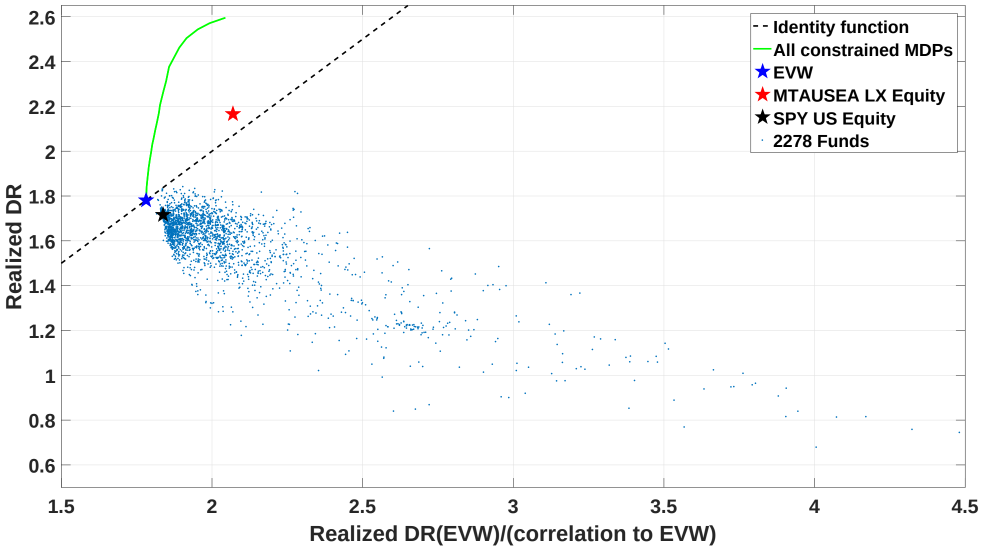

Figure 6: Realized DR and $DR(w_{evw})/\varrho(w_{evw}, \cdot)$ in the USA from 01/13 - 03/17 for 2278 funds representing half the total net assets invested in the USA in Q1/2016. The fund depicted with a blue star is a theoretical and forward looking EVW. The green curve depicts all the forward looking constrained MDPs. The green curve and the dashed line meet precisely at the forward-looking EVW portfolio. The fund in red is the `MOST DIV TOBAM A/B US EQ-A` that targets the highest investable DR whereas the fund in black replicates the S&P500. The blue dots depict all other funds. Only the portfolios that are above the dashed line - that depicts the identity - qualify for being maximally ρ-presentative as they satisfy the necessary condition $DR(w) \geq \frac{DR(w_{evw})}{\varrho(w, w_{evw})}$ of Proposition 4.5. The green curve corresponds to portfolios that are indeed maximally ρ-presentative by Proposition 4.15.

On a different topic, the fact that some funds have a DR that is less than one may indicate that they are not long-only or composed of assets that are outside of the considered universe. Indeed, as we do not have access to their compositions, we cannot guarantee that they are invested solely in the MSCI USA selection.

Finally, we refer to the Appendix for some additional illustrations of the theoretical results of this paper that are based on this dataset of funds.

6 Conclusion and Perspectives

As an alternative to portfolio weights, we introduced the equivalent representation offered by $\rho(w)$, the vector of correlations of a portfolio to all the assets of an investment universe. This new representation naturally leads to the notion of ρ-presentative portfolio - such as the ERC, MV and MDP - which allows an investor to be positively exposed to all assets without necessarily being invested in all of them.

We then complemented this notion by introducing the concept of maximally ρ-presentative portfolio, which maximizes its aggregated exposure to all assets. The real valued function f that measures the aggregated exposure of the portfolio is assumed to be symmetric, concave and increasing. We first proved that maximally ρ-presentative portfolios are long-only using a key lemma: for any portfolio that is not long-only, there always exists a long-only portfolio that is more correlated to all assets. A characterization of this new class of portfolios is then provided: its members are the long-only portfolios whose exposures form a non-increasing function of their volatility weighted weights. Well known members include the EVW, the ERC and MDPs that can be constrained with maximum weights.

As we have seen, the functions f provide a fairly general trade-off between the average, the dispersion and possibly higher moments of the exposures of a portfolio. Using Schur concave and increasing functions offers an avenue for further research to generalize the classic Mean-Variance approach to portfolio construction.

Having tackled the no-short sales constraints, we studied the impact of adding maximum weight constraints to the MDP and MV. The results provided in this paper extend the analytical results of Jagannathan and Ma (2003), as their impact on the original objective is made explicit and known *a priori*. Furthermore, in a context where the covariance matrix has to be estimated, this yields a plain interpretation of the impact of these constraints on the objective: reducing its estimation error.

We leave for further research the formal study of the biases and estimation variance reduction induced by the addition of constraints on the MDP and MV. It should be noted that even in a setting where returns are Gaussian, the problem is challenging as it depends on the order statistics of $\rho_{\hat{\Sigma}}(w^*(\hat{\Sigma}))$.

On another topic, many of the arguments in this paper (KKT, ball compactness, continuity of convex functions, etc.) rely on the fact that the analysis is performed in a finite-dimensional setting. It would be interesting to extend these results to a setting where there is a continuum of assets. In particular, we wonder what the set of maximally ρ-presentative portfolios would be in this case.

Also, the characterization of these rare portfolios that is at the core of this paper satisfies a purely algebraic property that deserves a more thorough analysis. Moreover, this problem seems to share connection with for instance task scheduling problems where the idle time is minimized under all permutations of the tasks.

Finally, our results are general as they only rely on the positivity of the correlation matrix and provide a unifying framework that encompasses many well-known and possibly constrained portfolios. Furthermore, beyond their financial implications, they may be useful in other fields where correlations are used to measure interactions.

Appendix

Proof of Proposition 2.5

Proof. The function ϕ is well defined on Π^+ as $\langle w, \sigma \rangle > 0$ over this set, the same goes for ϕ^{-1}, and it is easy to check that $\phi \circ \phi^{-1} = \phi^{-1} \circ \phi = I$, and as a result that $\langle w, \sigma \rangle \langle x, \mathbf{1} \oslash \sigma \rangle = 1$. Note that in the definition of ϕ, we simply need that the considered portfolios are not orthogonal to σ which, here, is implied by $\Sigma \succ 0$. Recalling that $D(\sigma)$ is the diagonal matrix with σ on the diagonal, one has $\Sigma = D(\sigma) C D(\sigma)$. Furthermore $\forall w \in \Pi^+, x = \phi(w) = \langle w, \sigma \rangle^{-1} D(\sigma) w$, so $D(\sigma) w = \langle x, \mathbf{1} \oslash \sigma \rangle^{-1} x$. Now, $\forall w_1, w_2 \in \Pi^+$,

$$\sigma_\Sigma^2(w_1) = \langle \Sigma w_1, w_1 \rangle = \langle C D(\sigma) w_1, D(\sigma) w_1 \rangle = \langle x_1, \mathbf{1} \oslash \sigma \rangle^{-2} \langle C x_1, x_1 \rangle, \text{ with } \langle x_1, \mathbf{1} \oslash \sigma \rangle > 0,$$

$$DR_\Sigma(w_1) = \frac{\langle w_1, \sigma \rangle}{\sigma_\Sigma(w_1)} = \frac{\langle x_1, \mathbf{1} \oslash \sigma \rangle}{\langle x_1, \mathbf{1} \oslash \sigma \rangle} \frac{1}{\sigma_C(x_1)} \geq 1 \text{ as } \sigma_C(x_1) \leq 1 \text{ since } x_1 \in \Pi^+,$$

$$\varrho_\Sigma(w_1, w_2) = \frac{\langle w_1, \Sigma w_2 \rangle}{\sigma_\Sigma(w_1)\sigma_\Sigma(w_2)} = \frac{\langle x_1, \mathbf{1} \oslash \sigma \rangle \langle x_2, \mathbf{1} \oslash \sigma \rangle}{\sigma_C(x_1)\sigma_C(x_2)} \langle D(\sigma) w_1, C D(\sigma) w_2 \rangle = \varrho_C(x_1, x_2).$$

Lastly, as $\phi(e_i) = e_i$, the last identity implies that $\rho_\Sigma(w_1) = \rho_C(x_1)$. □

Generalization of The Composition Formula in Proposition 2.4

Proposition. *If we have $m > 1$ portfolios $w_i \in \Pi$ that we arrange in columns in a $n \times m$ matrix W a $\theta \in \mathbb{R}^m$ with $\theta \succeq 0$ and $\langle \theta, \mathbf{1} \rangle = 1$, then*

$$\rho(W\theta) = d(W\theta)\rho(W)\Phi_{\sigma(w_i)_i}(\theta),$$

where

(i) $d(W\theta) = \frac{\langle \theta, [\sigma(w_i)]_i \rangle}{\sigma(W\theta)} \in [1, +\infty)$,

(ii) $\rho(W)$ *is the $n \times m$ matrix whose columns are the* $\rho(w_i)$,

(iii) $\Phi_{\sigma(w_i)_i}(\theta) = \frac{\theta \odot (\sigma(w_i)_i)}{\langle \theta, (\sigma(w_i)_i) \rangle} \in \mathbb{R}^m$ *and has nonnegative components that sum to one.*

If we take a k-homogeneous ($k > 0$) and concave $f : \mathbb{R}^n \to \mathbb{R}$ that we apply to the columns of $\rho(W)$ we have a property similar to strict convexity:

$$f \circ \rho(W\theta) \geq d(W\theta)^k [f \circ \rho(W)] \Phi_{\sigma(w_i)_i}(\theta).$$

If for any $w \in \Pi$, $\theta \in \Pi^+$, we consider the function $f(x) = \langle x, \frac{w}{\sigma(w)} \rangle$, $m := n$ and $W = Id$, then

$$\varrho(w, \theta) = DR(\theta)\langle \rho(w), \phi(\theta) \rangle.$$

The latter proposition generalizes Proposition 2.4 and relates it to the last statement of Lemma 3.4.

Realized RM_r

Let us observe that as for the realized DR, the realized RM_r of a portfolio (introduced in 4.3.1) may be measured without knowing its composition as the realized $\rho(w)_i$ is simply the correlation between the time series of w with that of asset i. Therefore, we can perform another numerical experimentsby placing ourselves in the same setting as 5.3 and considering the same 2278 funds and universe of 464 stocks. For each fund w and every integer $r \leq 464$, one can compute $RM_r(w)$ using the sample correlation. In Figure 7, we plot all the curves $r \mapsto RM_r(w)$ (that are non-decreasing by definition of RM).

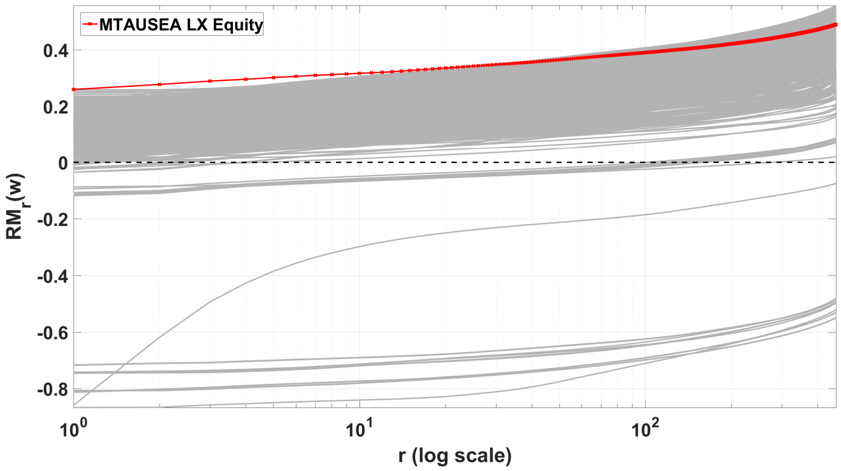

Figure 7: $(RM_r)_r$ in the USA for 2278 funds from 01/13 - 03/17. The red fund aims to maximize the DR.

For $r \leq 32$, the fund maximizing $RM_r(w)$ amongst all funds is the `MOST DIV TOBAM A/B US EQ-A` that targets the highest investable DR. Observe that $r = 32$ is the smallest integer such that this fund has not the highest $RM_r(w)$ suggesting that its *implicit volatility-adjusted maximum weight constraint* is larger than 3.13%.

By Proposition 5.3, we can also plot in Figure 8 the realized DR of all 2278 funds as a function of their RM_{32}. In addition, we plot in green the constrained MDPs computed over the whole window for all values of r. These are all forward looking portfolios. Observe that the unconstrained MDP is suboptimal in terms of RM_{32} whereas $DR(w_{32}^*) = RM_{32}(w_{32}^*)^{-1}$ as proven in Proposition 4.15.

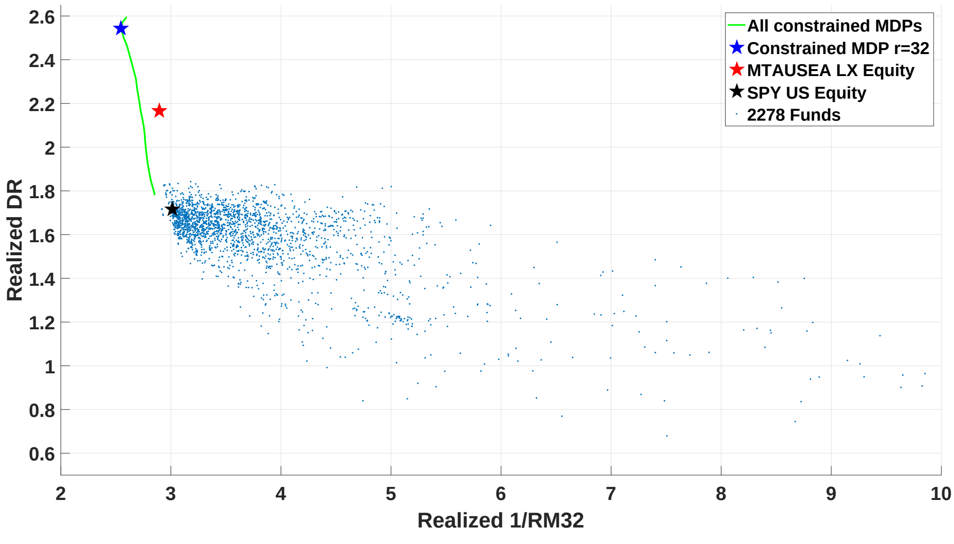

Figure 8: Realized DR and $1/RM_{32}$ in the USA from 01/13 - 03/17 for 2278 funds representing half the total net assets invested in the USA in Q1/2016. The fund depicted with a blue star is a theoretical and forward looking constrained MDP with r=32. The green curve depicts all the forward looking constrained MDPs. The fund in red is the `MOST DIV TOBAM A/B US EQ-A` that targets the highest investable DR whereas the fund in black replicates the S&P500 index. The blue dots depict all other funds.

Similarly, using for instance Proposition 3.5, we isolate in the list below the funds that may not be long-only. Their names clearly indicate that they are indeed all "short" or "bear" funds:

```
ADVISORSHARES RANGER EQ BEAR
DIREXION DAILY FINL BEAR 3X
DIREXION DAILY S&P 500 BEAR
DIREXION DLY SM CAP BEAR 3X
GRIZZLY SHORT FUND
PROSH ULTRAPRO SHORT S&P 500
PROSHARES SHORT DOW30
PROSHARES SHORT QQQ
PROSHARES SHORT RUSSELL2000
PROSHARES SHORT S&P500
PROSHARES ULTPRO SHRT DOW30
PROSHARES ULTRAPRO SHORT QQQ
PROSHARES ULTRAPRO SHRT R2K
PROSHARES ULTRASHORT DOW30
PROSHARES ULTRASHORT QQQ
PROSHARES ULTRASHORT R2000
PROSHARES ULTRASHORT S&P500
```

One could have also used inequality (4.19) or the second assertion of Lemma 4.3 to isolate funds that may not be long-only and thus not maximally ρ-presentative.

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