Staying at the Zero Lower Bound with Embedded Markov Chain

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Abstract

This paper introduces a new dynamic modelling of the Zero Lower Bound (ZLB) phenomenon based on the notion of Embedded Markov Chain. The model provides (quasi) closed-form expressions for the term structure of interest rates as well as for the price of European and Asian type derivatives written on the rates. The model is flexible since the underlying unobservable factors of the term structure can be specific to either the term structure in the ZLB regime, or the term structure in the non-ZLB state. These properties are illustrated numerically.

Keywords: Endogenous Regime Switching, Zero Lower Bound, Term Structure, Lift-Off Probability, Embedded Markov Chain, Economic Scenario Generator.

JEL code: C32, C14

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1 Introduction

The standard term structure models are usually affine models constructed to avoid arbitrage opportunities on the fixed income markets and to ensure strictly positive rates. Typical examples

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include the Cox-Ingersoll-Ross model [Cox et al. (1985)] and its multivariate extensions [see e.g. Dai and Singleton (2000); Ait-Sahalia and Kimmel (2010)]. Their application on series with observed zero rate [see e.g. Swanson and Williams (2014)] can generate implausible nominal risk premia [see Kim and Singleton (2012)] and imprecise long term predictions.

Indeed, the strictly positive feature of the nominal short term interest rate in these standard models is not compatible with the zero (or near zero) rates recently observed in several countries, since 1996 for the Japanese Government bonds, after the 2008 financial crisis for the U.S. Treasury bills and more recently in France[1]. This is the so-called Zero-Lower-Bound (ZLB) phenomenon for the short term rate.

Despite the importance of this ZLB phenomenon for bond pricing, risk management, or macroeconomic and monetary policies, and the growing literature on this latter topic[2], the dynamic modelling of the ZLB phenomenon is still in its infancy. Loosely speaking, three types of dynamic models have been considered either in continuous, or discrete time.

i) The shadow (short) rate model (SRM) has been initially introduced by Black (1995) and Rogers (1995), then used in a number of academic papers [see e.g. Kim and Singleton (2012); Ichiue and Ueno (2013); Swanson and Williams (2014); Imakubo and Nakajima (2015); Christensen and Rudebusch (2014, 2015)]. The basic idea is the following: a Gaussian affine model is introduced to define the dynamics of underlying factors X_t, say, usually three factors interpreted as level, slope and curvature factors. Then a shadow short term interest rate r_t^* is defined as a linear combination $r_t^* = \alpha' X_t$ of these factors. In such a Gaussian affine model, the shadow rate can take positive as well as negative values. Then the observed short term rate is defined[3] as $r_t = \max(r_t^*, 0)$. The SRM does not allow for closed form pricing formula for zero coupon bonds.

ii) An alternative approach has been introduced in Monfort et al. (2017). They show that it is possible to construct in discrete time purely affine term structure models, that can stay at zero during endogenous periods. The modelling is based on a limiting case of Autoregressive Gamma (ARG) process[4], called ARG-zero process, used to describe some factor dynamics.

[1]Japan has been confronted with extremely low rates since the mid-90s, the Federal Reserve lowered its prime rate (i.e. the federal fund rate) to almost zero in December 2008, the Bank of England in early 2009, and the German rate is near zero at the beginning of 2012. Zero lower bound has also been observed in the 1930's in US. The discussion of strictly negative short term rates observed for France and Germany is out of the scope of this paper.

[2]Under near zero rates, the central banks have no room for further monetary easing policy by lowering their prime rate. The shadow rate model has been applied by Central Banks to find alternative nonstandard monetary policies [see e.g. Hamilton and Wu (2012); Bauer and Rudebusch (2016); Wu and Xia (2016); Alevskis (2016); Deutsche Bundesbank (2017)].

[3]When a central bank pays interest on excess reserves, $\underline{r_t}$, say, the censored value could be defined as $r_t = \max(r_t^*, \underline{r_t})$. Since $\underline{r_t}$ is generally very small, we set it to $\underline{r_t} = 0$, as usually done in the literature.

[4]An ARG process is the exact time discretized Cox-Ingersoll-Ross process [see Gouriéroux and Jasiak (2006)].

iii) Finally Christensen (2015) has considered a 4-factor model with the first 3 factors for the level, slope and curvature, and the additional factor to represent the stochastic intensity of exit from the ZLB state. By taking partly into account the regime switching between the zero and normal states, this model outperforms the SRM for an identical number of factors[5].

However these different modellings do not seem flexible enough. Clearly Christensen's approach assumes that the dynamics of level, slope and curvature is the same in the ZLB and the non ZLB state, and nothing is said about the possibility of reverting to the ZLB after an exit.

In the other two modellings, any underlying factor will have a joint effect on the term structure in the ZLB state, the term structure in the non-ZLB state, the intensity of exiting the ZLB and the intensity of entering in the ZLB.

The aim of our paper is to propose a new modelling in which some underlying factors can be specific to either the term structure in the ZLB state, or specific to the term structure in the non-ZLB state. The model is based on Markov processes with Embedded Markov Chains (EMC), which offer closed form expressions of the zero-coupon bond prices and of various interest rate derivatives, such as swaps, swaptions, and caplets.

The remainder of this paper is organized as follows. In Section 2 we introduce the general concept of Markov process with Embedded Markov Chain, and show how it can be used to account for the zero lower bound phenomenon for the short term interest rate. Section 3 discusses the duration of a spell at the ZLB (resp. in the non-ZLB state) and the total time spent at the ZLB. Section 4 derives the pricing formulas for the zero-coupon bonds. Section 5 proposes an estimation strategy. The finite sample properties of the estimators are analyzed by Monte-Carlo and the method is applied to real data. Section 6 concludes. Proofs are provided in Appendices.

2 The model

2.1 The Markov model with Embedded Markov Chain (EMC)

To define a simple, yet flexible, dynamics of a (multivariate) Markov process (X_t), it is convenient to introduce an underlying Markov chain (Z_t) with finite state space $\{1, ..., K\}$, according to the following causal scheme:

[5]A switching regime model is also considered in Hördahl and Tristani (2018), but with an unobservable stochastic lower bound. This modelling cannot capture the flatness of rate history during some endogenous spells.

$$X_{t-1} \to Z_t \to X_t \to Z_{t+1} \to X_{t+1}.$$

Later on, $\underline{X_t}$ (resp. $\underline{Z_t}$) denotes the information included in the current and past values of process X (resp. Z). For instance, $\underline{X_t} = (X_t, X_{t-1}, X_{t-2}, ...)$.

Then we say that:

Definition 1. The process (X_t) is Markov with EMC (Z_t) *iif*,

i) the conditional distribution of X_t given $\underline{Z_t}$, $\underline{X_{t-1}}$ depends on the past through Z_t only;

ii) the conditional distribution of Z_t given $\underline{Z_{t-1}}$, $\underline{X_{t-1}}$ depends on the past through X_{t-1} only,

By iterated conditioning, it is easily checked that the process (X_t) [resp. (Z_t)] is Markov with respect to its own sequence of information sets $\underline{X_{t-1}}$ (resp. $\underline{Z_{t-1}}$).

The joint dynamics of these two processes is defined by:

- the conditional distribution of Z_{t+1} given X_t, and
- the conditional distribution of X_{t+1} given Z_{t+1}.

The first conditional distribution is characterized by the vector $\beta(x_t)$ of the K elementary conditional probabilities:

$$\beta_k(x_t) = \mathbb{P}[Z_{t+1} = k | X_t = x_t], \qquad k = 1, ..., K, \tag{2.1}$$

with $\beta_k(x_t) > 0, k = 1, ..., K$, $\sum_{k=1}^{K} \beta_k(x_t) = 1$.

Next, the second conditional distribution is characterized by a set of conditional densities $\alpha_k(x_{t+1})$ with respect to a common dominating measure μ, say, when $Z_{t+1} = k$, for $k = 1, ..., K$, with $\alpha_k(x) \geq 0$, and $\int \alpha_k(x) \mathrm{d}\mu(x) = 1$, for any $k = 1, ..., K$. They can be stacked in a $K-$ dimensional vector function $\alpha(x_{t+1})$.

The transition density of process (X_t) has the form of a mixture with path dependent weights:

$$f_1(x_{t+1}|x_t) = \beta'(x_t)\alpha(x_{t+1}), \tag{2.2}$$

and the transition matrix of the EMC, denoted by $\Pi = (\pi_{k,l}) = (\mathbb{P}[Z_t = l | Z_{t-1} = k])$, is equal to:

$$\Pi = \int \alpha(x)\beta'(x) \mathrm{d}\mu(x), \tag{2.3}$$

by applying the Bayes formula. This EMC structure facilitates the nonlinear prediction of the process (X_t) at any horizon:

Proposition 1 (see also Gouriéroux and Jasiak (2001)). *The transition density of the Markov process (X_t) at horizon h is:*

$$f_h(x_{t+h}|x_t) = \beta'(x_t)\Pi^{h-1}\alpha(x_{t+h}). \tag{2.4}$$

Proof. Indeed we get:

$$\begin{aligned} f_h(x_{t+h}|x_t) &= \iint \beta'(x_t)\alpha(x_{t+1})\beta'(x_{t+1})\cdots\beta'(x_{t+h-1})\alpha(x_{t+h})\mathrm{d}x_{t+1}\cdots\mathrm{d}x_{t+h-1} \\ &= \beta'(x_t)\Pi^{h-1}\alpha(x_{t+h}) \end{aligned}$$

□

Π is a transition matrix, that is, this matrix has strictly positive entries and each of its rows sums up to unity. Thus Π admits 1 as eigenvalue, and by Perron-Frobenius theorem [see Nummelin (1978)], all the other eigenvalues of Π have a modulus strictly smaller than one. Then the vector $\beta'(x_t)\Pi^{h-1}$ tends to a row vector v' with strictly positive components, when horizon h goes to infinity, and v is the unique normalized left eigenvector of Π associated with eigenvalue 1. We deduce the ergodicity of the EMC process:

Proposition 2. *The Markov process with EMC defined in Proposition 1 is ergodic and its stationary density is $f(x_t) = v'\alpha(x_t)$, where v is the normalized left eigenvector of Π associated with eigenvalue 1.*

2.2 A dynamics with Zero Lower Bound

The Markov process with EMC can be used to model the dynamics of a short term interest rate with Zero Lower Bound (ZLB). Let us consider the process $X_t = (r_t, Y_t')'$, where the first component is the riskfree short term interest rate and the p other components are additional factors with potential effects on the term structure and on its dynamics. To particularize the ZLB, we assume that the marginal and conditional distributions of the rate r_t are mixtures of a point mass at zero and of a continuous component on the positive real half-line, whereas the components of Y_t are continuous. Therefore the dominating measure is $\mu = (\delta_0 + \lambda^+) \otimes \lambda^p$, where δ_0 denotes the point mass at zero, λ^+ and λ^p are the Lebesgue measures on $]0, \infty[$ and $]-\infty, \infty[^p$, respectively.

The embedded process (Z_t) is linked to the interest rate by assuming that $Z_t = (\mathbb{1}_{r_t>0}, S_t)$, where S_t is a state variable with finite state space $\{1, ..., S\}$. Thus the Markov chain (Z_t) has $K = 2S$ possible states $(0, s), (1, s), s = 1, ..., S$, and the causal scheme becomes:

$$(r_{t-1}, Y_{t-1}) \longrightarrow (\mathbb{1}_{r_t>0}, S_t) \longrightarrow (r_t, Y_t) \longrightarrow (\mathbb{1}_{r_{t+1}>0}, S_{t+1}) \longrightarrow (r_{t+1}, Y_{t+1}). \tag{2.5}$$

The deterministic relationship between r_t and the first component of Z_t implies new interpretations of the functions α and β introduced in the previous subsection.

i) **Conditional distribution of Z_{t+1} given X_t.**

The definition (2.1) becomes:

$$\beta_{0,s}(x_t) = \mathbb{P}[r_{t+1} = 0, S_{t+1} = s | X_t = x_t], \tag{2.6}$$

$$\beta_{1,s}(x_t) = \mathbb{P}[r_{t+1} > 0, S_{t+1} = s | X_t = x_t]. \tag{2.7}$$

These quantities can be stacked into vectors as:

$$\beta_0(x_t) = \left(\beta_{0,s}(x_t)\right)_{s=1,\ldots,S}, \qquad \beta_1(x_t) = \left(\beta_{1,s}(x_t)\right)_{s=1,\ldots,S}, \qquad \beta(x_t) = \left(\beta_0'(x_t), \beta_1'(x_t)\right)'.$$

They satisfy the constraint:

$$\beta_0(x_t)'\mathbb{1}_S + \beta_1(x_t)'\mathbb{1}_S = 1,$$

where $\mathbb{1}_S$ is the S dimensional vector with unitary components. These probabilities can be used to deduce the probability of being at the ZLB (resp. non-ZLB) in the next period:

$$p(x_t) := \mathbb{P}[r_{t+1} = 0 | X_t = x_t] = \sum_{s=1}^{S} \beta_{0,s}(x_t) = \beta_0(x_t)'\mathbb{1}_S, \tag{2.8}$$

$$1 - p(x_t) := \mathbb{P}[r_{t+1} > 0 | X_t = x_t] = \beta_1(x_t)'\mathbb{1}_S, \tag{2.9}$$

as well as the conditional probability distribution of the second regime variable S_{t+1} at the ZLB (resp. non-ZLB):

$$\gamma_{0,s}(x_t) := \mathbb{P}[S_{t+1} = s | r_{t+1} = 0, X_t = x_t] = \frac{\beta_{0,s}(x_t)}{\beta_0(x_t)'\mathbb{1}_S}, \qquad s = 1, ..., S, \tag{2.10}$$

$$\gamma_{1,s}(x_t) := \mathbb{P}[S_{t+1} = s | r_{t+1} > 0, X_t = x_t] = \frac{\beta_{1,s}(x_t)}{\beta_1(x_t)'\mathbb{1}_S}, \qquad s = 1, ..., S. \tag{2.11}$$

In other words the conditional distribution of Z_{t+1} given X_t can be equivalently characterized by β_0, β_1, or by γ_0, γ_1 and π. Both characterizations are used later on.

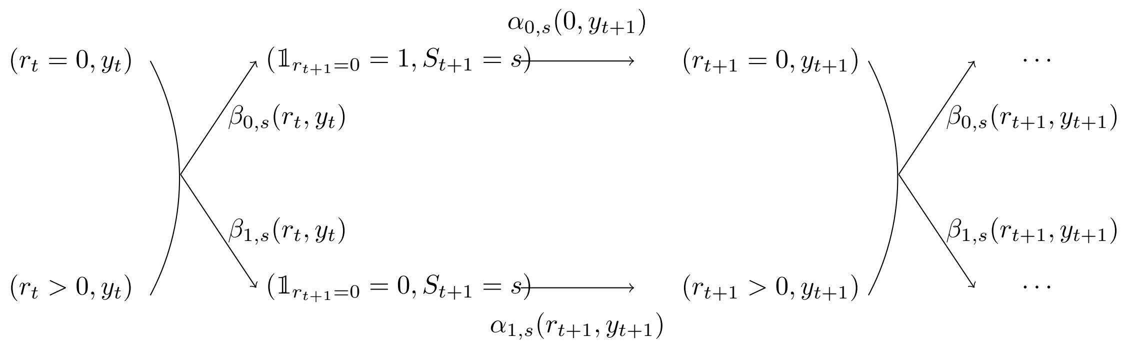
ii) **Conditional distribution of X_{t+1} given Z_{t+1}.**

The conditional densities with respect to μ are now denoted $\alpha_{0,s}(x_{t+1})$ and $\alpha_{1,s}(x_{t+1})$ for $Z_{t+1} = (0, s)$ and $Z_{t+1} = (1, s)$, respectively.

Conditional on $Z_{t+1} = (0, s)$, we know that $X_{1,t+1} = r_{t+1} = 0$; therefore $\alpha_{0s}(x_{t+1}) = \mathbb{1}_{r_{t+1}=0}\alpha_{0,s}(0, y_{t+1})$, where $\alpha_{0,s}(0, y_{t+1})$ is the density of Y_{t+1} given $X_{1,t+1} = r_{t+1} = 0$ and $S_{t+1} = s$.

Conditional on $Z_{t+1} = (1, s)$, $\alpha_{1,s}(x_{t+1}) = \alpha_{1s}(r_{t+1}, y_{t+1})$ is the joint density of (r_{t+1}, y_{t+1}) given $X_{1,t+1} = r_{t+1} > 0$ and $S_{t+1} = s$.

To summarize, the causal chain (2.5) can be written in a more detailed form:



In other words we get a two-layer factor model. The first layer is characterized by the discrete state variable S_{t+1}, then the second layer depends on the quantitative factor Y_{t+1}. In the next section we use the terminology state variable for S_{t+1}, factor for Y_{t+1}.

As a consequence, the transition of (r_{t+1}, y_{t+1}) given its past has a density with respect to measure μ given by:

$$f(r_{t+1}, y_{t+1} | r_t, y_t) = \beta'(r_t, y_t)\alpha(r_{t+1}, y_{t+1}) = \beta_0'(r_t, y_t)\alpha_0(0, y_{t+1})\mathbb{1}_{r_{t+1}=0} + \beta_1'(r_t, y_t)\alpha_1(r_{t+1}, y_{t+1})\mathbb{1}_{r_{t+1}>0}, \tag{2.12}$$

where $\alpha(x_{t+1}) = (\alpha_0(x_t)', \alpha_1(x_t)')'$. Thus, by Proposition 1, we get:

Corollary 1. *The term structure of nonlinear forecasts is:*

$$f(r_{t+h}, y_{t+h}|r_t, y_t) = \beta'(r_t, y_t)\Pi^{h-1}\alpha(r_{t+h}, y_{t+h}), \quad \forall h \geq 1,$$

$$\begin{aligned} \textit{where } \Pi &= \int \alpha(r,y)\beta'(r,y)\mathrm{d}\mu(r,y) \\ &= \begin{bmatrix} \int \alpha_0(0,y)\beta_0'(0,y)\mathrm{d}y & \int \alpha_0(0,y)\beta_1'(0,y)\mathrm{d}y \\ \int \alpha_1(r,y)\beta_0'(r,y)\mathrm{d}\mu(r,y) & \int \alpha_1(r,y)\beta_1'(r,y)\mathrm{d}\mu(r,y) \end{bmatrix} := \begin{bmatrix} \Pi_{00} & \Pi_{01} \\ \Pi_{10} & \Pi_{11} \end{bmatrix}, \end{aligned} \tag{2.13}$$

is the $(2S \times 2S)$ transition matrix of the chain (Z_t), and the conditional probability that r_{t+h} is equal to zero is:

$$\mathbb{P}[r_{t+h} = 0|r_t, y_t] = \beta'(r_t, y_t)\Pi^{h-1}\begin{bmatrix} \mathbb{1}_S \\ 0_S \end{bmatrix}, \qquad \forall h \geq 1. \tag{2.14}$$

where 0_S is the $S-$dimensional null vector.

The following corollary is a consequence of Proposition 2.

Corollary 2. *i) The process (X_t) is stationary and ergodic.*

ii) The stationary distribution of the process $X_t = (r_t, y_t)$ is given by $v'\alpha(r_t, y_t)$, where v is the vector of stationary distribution of the Markov chain (Z_t), that is the left normalized unitary eigenvector of Π, defined by $v'\Pi = v'$; $v'\mathbb{1}_{2S} = 1$. In particular, the marginal probability of $r_t = 0$ is the sum of the S first components of vector v, that is $v'\mathbb{1}_S$.

Example 1. Let us consider the special case where $p(x_t) = p_1$, if $r_t = 0$, and $p(x_t) = p_2$, otherwise. Then Π becomes:

$$\Pi = \begin{bmatrix} p_1\tilde{\Pi}_{00} & (1-p_1)\tilde{\Pi}_{01} \\ p_2\tilde{\Pi}_{10} & (1-p_2)\tilde{\Pi}_{11} \end{bmatrix}, \tag{2.15}$$

where $\tilde{\Pi}_{i,j}, i,j = 0,1$ are lower-dimensional stochastic transition matrices. For instance $\tilde{\Pi}_{00} = \int \alpha_0(0,y)\gamma_0'(0,y)\mathrm{d}y$. The interpretation of decomposition (2.15) is the following: within a realized spell at (resp. outside) the ZLB, the posterior dynamics of S_t is Markov with transition matrix $\tilde{\Pi}_{00}$ (resp. $\tilde{\Pi}_{11}$), whereas the dynamics of the indicator $\mathbb{1}_{r_t>0}$ is a Markov chain with transition matrix $\begin{bmatrix} p_1 & 1-p_1 \\ p_2 & 1-p_2 \end{bmatrix}$. In particular, the serial dependence of the different components of X_t

mainly depends on the second largest eigenvalue of $\tilde{\Pi}_{00}$ (resp. $\tilde{\Pi}_{11}$) while at (resp. outside) the ZLB. Similarly, $\tilde{\Pi}_{10}$ (resp. $\tilde{\Pi}_{01}$) is the transition matrix of (S_t), when the regime variable $\mathbb{1}_{r_t \geq 0}$ moves from ZLB (resp. non-ZLB) to non-ZLB (resp. ZLB) state. In the general case, where $p(x_t)$ is not constant, $\tilde{\Pi}_{10}$ (resp. $\tilde{\Pi}_{01}$) is no longer the product between a probability and a transition matrix. However, the interpretation above is still useful in understanding the duration analysis of the next section.

3 Duration analysis

Let us now focus on the total (stochastic) time spent at the ZLB and on the (stochastic) duration of a spell at the ZLB (resp. at the non-ZLB state).

3.1 Staying at the Zero Lower Bound

Let us consider a date t at which the short term interest rate is at the ZLB with factor value y_t. The distribution of the residual duration spent at the ZLB before lifting-off is characterized by the sequence of cumulative survival probabilities:

$$\mathcal{S}_{00}(h, y_t) = \mathbb{P}[r_{t+h} = r_{t+h-1} = \cdots = r_{t+1} = 0 | r_t = 0, Y_t = y_t], \qquad h \geq 1,$$

or equivalently by the instantaneous survival probabilities:

$$p_{00}(h, y_t) = \mathbb{P}[r_{t+h} = 0 | r_{t+h-1} = \cdots = r_t = 0, Y_t = y_t] = \frac{\mathcal{S}_{00}(h, y_t)}{\mathcal{S}_{00}(h-1, y_t)}, \qquad h \geq 1,$$

with the convention $\mathcal{S}_{00}(0, y_t) = 1$.

The complements to 1 of these instantaneous survival probabilities, that are $1 - p_{00}(h, y_t)$, are the lift-off probabilities. Note that the available information y_t is the information at date t; this information does not vary with horizon h and the forward lift-off probabilities $1 - p_{00}(h, y_t)$ usually differ from their spot counterparts that are $1 - p_{00}(1, y_{t+h-1})$, except when $h = 1$. In this latter case, we have: $p_{00}(1, y_t) = \mathcal{S}_{00}(1, y_t) = p(0, y_t)$, which is the spot survival probability. In the following, we call $p_{00}(h, y_t)$, where $h \geq 2$, the forward instantaneous survival probability, and we are interested in their term structure.

Proposition 3. *We have,*

$$\mathcal{S}_{00}(h, y_t) = \beta_0'(0, y_t)\Pi_{00}^{h-1}\mathbb{1}_S, \tag{3.1}$$

where Π_{00} *is defined in equation* (2.13)*.*

Proof. See Appendix 1.1. □

As a consequence, the forward instantaneous survival probabilities are equal to:

$$p_{00}(h, y_t) = \frac{\beta_0'(0, y_t)\Pi_{00}^{h-1}\mathbb{1}_S}{\beta_0'(0, y_t)\Pi_{00}^{h-2}\mathbb{1}_S} = \frac{\gamma_0'(0, y_t)p(0, y_t)\Pi_{00}^{h-1}\mathbb{1}_S}{\gamma_0'(0, y_t)p(0, y_t)\Pi_{00}^{h-2}\mathbb{1}_S} = \frac{\gamma_0'(0, y_t)\Pi_{00}^{h-1}\mathbb{1}_S}{\gamma_0'(0, y_t)\Pi_{00}^{h-2}\mathbb{1}_S}, \qquad h \geq 2. \tag{3.2}$$

At horizons $h \geq 2$, the forward instantaneous survival probabilities depend only on the value of $\gamma_0(0, y_t)$, but not on $p(0, y_t)$.

Remark 1. Let us re-consider Example 1 introduced in Section 2. When $\Pi_{00} = p_1\tilde{\Pi}_{00}$, Proposition 3 becomes:

$$\begin{aligned} \mathcal{S}_{00}(h, y_t) &= p_1^{h-1}\beta_0'(0, y_t)\tilde{\Pi}_{00}^{h-1}\mathbb{1}_S \\ &= p_1^{h-1}\beta_0'(0, y_t)\mathbb{1}_S = p_1^{h-1}p_1 = p_1^h, \end{aligned}$$

where we have used the fact that $\tilde{\Pi}_{00}\mathbb{1}_S$ is a transition matrix under the assumptions of Example 1. This result is quite intuitive, since, when the transition probability p_1 of the regime variable $\mathbb{1}_{r_t>0}$ is constant, the probability of staying at the ZLB for h consecutive periods is simply p_1^h.

3.2 Staying above the Zero Lower Bound

Let us now assume that $r_t > 0$ at a certain date t. To analyse the duration of a spell at strictly positive rates, we consider the cumulative survival probability:

$$\mathcal{S}_{11}(h, r_t, y_t) = \mathbb{P}[r_{t+h} > 0, r_{t+h-1} > 0, \cdots, r_{t+1} > 0 | r_t, y_t], \quad h \geq 1, \tag{3.3}$$

for $r_t > 0$. These cumulative survival probabilities can also be characterized by a set of forward instantaneous survival probabilities:

$$p_{11}(h, r_t, y_t) = \mathbb{P}[r_{t+h} > 0 | r_{t+h-1} > 0, \cdots, r_{t+1} > 0, r_t, y_t] = \frac{\mathcal{S}_{11}(h, r_t, y_t)}{\mathcal{S}_{11}(h-1, r_t, y_t)}, \quad h \geq 1. \tag{3.4}$$

They now depend on the levels of both r_t and y_t. The following proposition is analogous to Proposition 3:

Proposition 4. *We have:*

$$\mathcal{S}_{11}(h, r_t, y_t) = \beta_1'(r_t, y_t)\Pi_{11}^{h-1}\mathbb{1}_S, \tag{3.5}$$

where Π_{11} *is defined in equation* (2.13)*.*

The proof is similar to that of the previous proposition and is omitted. Again, similar as in Remark 1, under the assumption of Example 1, this proposition reduces to:

$$\mathcal{S}_{11}(h, r_t, y_t) = p_2^h,$$

whereas the forward instantaneous survival probabilities are given by:

$$p_{11}(1, r_t, y_t) = \mathcal{S}_{11}(1, r_t, y_t) = 1 - p(r_t, y_t), \tag{3.6}$$

$$p_{11}(h, r_t, y_t) = \frac{[1 - p(r_t, y_t)]\gamma_1'(r_t, y_t)\Pi_{11}^{h-1}\mathbb{1}_S}{[1 - p(r_t, y_t)]\gamma_1'(r_t, y_t)\Pi_{11}^{h-2}\mathbb{1}_S} = \frac{\gamma_1'(r_t, y_t)\Pi_{11}^{h-1}\mathbb{1}_S}{\gamma_1'(r_t, y_t)\Pi_{11}^{h-2}\mathbb{1}_S}, \qquad \forall h \geq 2. \tag{3.7}$$

The term structures of forward instantaneous survival probabilities $p_{11}(h, r_t, y_t)$ and $p_{00}(h, y_t)$ are governed by two separate sets of functions $\gamma_0(0, y_t)$ and $\gamma_1(r_t, y_t)$, respectively. This explains the greater flexibility of our model. For instance in the ARG-zero model [see Monfort et al. (2017)], these two term structures are driven by a same set of factors.

3.3 Total time spent at the Zero Lower Bound

To find the distribution of the total time spent by the short term interest rate at the ZLB between t and $t + h$, let us introduce the random variable:

$$D(t, h) := \mathbb{1}_{r_{t+1}=0} + \cdots + \mathbb{1}_{r_{t+h}=0}.$$

This total time takes into account the possibility of several spells during the next h periods.

First, the conditional expectation $\mathbb{E}[D(t, h)|r_t, y_t]$ can be obtained by using directly equation (2.14):

$$\mathbb{E}[D(t, h)|r_t, y_t] = \sum_{k=1}^{h} \mathbb{P}[r_{t+k} = 0|r_t, y_t] = \beta'(r_t, y_t) \sum_{k=1}^{h} \Pi^{k-1} \begin{bmatrix} \mathbb{1}_S \\ 0_S \end{bmatrix}. \tag{3.8}$$

In particular the long-run average time spent at the ZLB state, i.e. $\lim_{h\infty} \frac{1}{h}\mathbb{E}[D(t,h)|r_t, y_t]$, is equal to $v' \begin{bmatrix} \mathbb{1}_S \\ 0_S \end{bmatrix}$, where v is defined in Corollary 2.

Next, in order to get the predictive distribution of $D(t,h)$, let us consider the conditional Laplace transform of $D(t,h)$:

$$\tau(t,h,u) = \mathbb{E}[e^{-uD(t,h)}|r_t, y_t], \qquad u > 0.$$

Proposition 5. *For each $h \geq 1$, we have:*

$$\tau(t,h,u) = \beta'(r_t, y_t) M^{h-1}(u) \begin{bmatrix} \exp(-u)\mathbb{1}_S \\ \mathbb{1}_S \end{bmatrix}, \tag{3.9}$$

where the $(2S \times 2S)$ matrix function $M(u)$ is given by:

$$M(u) = \int \exp(-u\mathbb{1}_{r=0})\alpha(r,y)\beta'(r,y)\mathrm{d}\mu(r,y) = \begin{bmatrix} \exp(-u)\Pi_{00} & \exp(-u)\Pi_{01} \\ \Pi_{10} & \Pi_{11} \end{bmatrix}.$$

Proof. See Appendix 1.2. □

This result can be used to compute the conditional probability mass function (p.m.f.) of the discrete variable $D(t,h)$. Indeed, we have:

$$\mathbb{E}[e^{-uD(t,h)}|r_t, y_t] = \sum_{k=0}^{h} e^{-ku}\mathbb{P}[D(t,h) = k|r_t, y_t],$$

which is a polynomial in e^{-u}. Its coefficients, that are the conditional p.m.f., can be computed by expanding the RHS of (3.9) in e^{-u}. This can be conducted using a symbolic calculation package such as Mathematica.

4 Pricing

To analyse the future risk of a portfolio including bonds and/or interest rate derivatives such as swaps, swaptions, or caplets, it is necessary to model in a coherent way the historical and risk-neutral dynamics, the latter one being adjusted for risk premia and used for derivative pricing.

This joint modelling is deduced from the specification of the historical dynamics (see Sections 2-3) and a stochastic discount factor (sdf).

4.1 Risk-neutral dynamics

Let us now specify the sdf. The sdf m_{t+1} between dates t and $t+1$ satisfies the no-arbitrage condition:

$$\mathbb{E}[m_{t+1}|r_t, y_t] = \exp(-r_t). \tag{4.1}$$

A specification compatible with the above condition is proposed below:

Assumption 1. *The sdf is defined by:*

$$m_{t+1} = \frac{\exp(-r_t)\kappa(r_{t+1}, y_{t+1})}{\mathbb{E}[\kappa(r_{t+1}, y_{t+1})|r_t, y_t]} = \frac{\exp(-r_t)\kappa(r_{t+1}, y_{t+1})}{\beta'(r_t, y_t)\int \kappa\alpha}, \tag{4.2}$$

where $\kappa(r_{t+1}, y_{t+1})$ *is a positive scalar function and* $\int \kappa\alpha$ *is the abbreviation of the vector* $\int \kappa(r, y)\alpha(r, y)\mathrm{d}\mu(r, y)$.

This specification is constrained since function κ depends on r_{t+1}, y_{t+1} only, but not on past values $\underline{r_t}$, $\underline{y_t}$. Then the sdf depends on $r_{t+1}, y_{t+1}, r_t, y_t$. For instance, one possible parametric form of the function κ is given by:

$$\kappa(r_{t+1}, y_{t+1}) = \exp(d_1 r_{t+1} + d_2' y_{t+1}), \tag{4.3}$$

where d_1 is scalar and d_2 has the same dimension as y_t. This sdf has the standard exponential affine form with respect to r_{t+1}, y_{t+1}, with an appropriate path-dependent drift in the log-sdf.

Corollary 3. *The risk-neutral conditional density of process* (r_t, y_t) *is:*

$$\begin{aligned} f^*(r_{t+1}, y_{t+1}|r_t, y_t) &= \frac{m_{t+1} f(r_{t+1}, y_{t+1}|r_t, y_t)}{\int m_{t+1} f(r_{t+1}, y_{t+1}|r_t, y_t)\mathrm{d}\mu(r_{t+1}, y_{t+1})} \\ &= \frac{\kappa(r_{t+1}, y_{t+1})\beta'(r_t, y_t)\alpha(r_{t+1}, y_{t+1})}{\beta'(r_t, y_t)\int \kappa\alpha}. \end{aligned} \tag{4.4}$$

Thus, as the historical dynamics, the risk-neutral dynamics of the joint process (r_t, y_t) can still be decomposed as:

$$f^*(r_{t+1}, y_{t+1}|r_t, y_t) = [\beta^*(r_t, y_t)]'\alpha^*(r_{t+1}, y_{t+1}), \tag{4.5}$$

with:

$$\alpha^*_{i,s}(r_{t+1}, y_{t+1}) = \frac{\kappa(r_{t+1}, y_{t+1})\alpha_{i,s}(r_{t+1}, y_{t+1})}{\int \kappa\alpha_{i,s}}, \tag{4.6}$$

$$\beta^*_{i,s}(r_t, y_t) = \frac{\beta_{i,s}(r_t, y_t)\int \kappa\alpha_{i,s}}{\beta'(r_t, y_t)\int \kappa\alpha}, \qquad \forall i = 0, 1, s = 1, ..., S. \tag{4.7}$$

In representation (4.5), the components of α^* are still joint densities, whereas β^* is still a vector of (risk-neutral) probabilities. Therefore, process (r_t, y_t) is Markov with EMC under both the historical and risk-neutral dynamics with the same underlying finite state space.

Let us now write the counterparts of equations (2.6) to (2.11) under the risk-neutral dynamics. Function β^* can be written as:

$$\beta^*(r_t, y_t) = (\beta^*_0(r_t, y_t)', \beta^*_1(r_t, y_t)')', \tag{4.8}$$

$$\text{with:} \qquad \beta^*_0(r_t, y_t) = p^*(r_t, y_t)\gamma^*_0(y_t), \tag{4.9}$$

$$\beta^*_1(r_t, y_t) = [1 - p^*(r_t, y_t)]\gamma^*_1(r_t, y_t), \tag{4.10}$$

where the $S-$dimensional functions γ^*_0 and γ^*_1 are given by:

$$\gamma^*_i(r_t, y_t) = \frac{\beta^*_i(r_t, y_t)}{\beta^*_i(r_t, y_t)'\mathbb{1}_S}, \qquad i = 0, 1. \tag{4.11}$$

By equation (4.10) we can now plug the historical betas in the expression above to get:

$$\gamma^*_{i,s}(r_t, y_t) = \frac{\beta_{i,s}(r_t, y_t)\int \kappa\alpha_{i,s}}{\sum_{i=1}^S \beta_{i,s}(r_t, y_t)\int \kappa\alpha_{i,s}} = \frac{\gamma_{i,s}(r_t, y_t)\int \kappa\alpha_{i,s}}{\sum_{i=1}^S \gamma_{i,s}(r_t, y_t)\int \kappa\alpha_{i,s}}, \qquad i = 0, 1, s = 1, ..., S.$$

Then the risk-neutral probability p^* is obtained by summing the S first (resp. last) components of function $\beta^*(r_t, y_t)$:

$$\begin{aligned} p^*(r_t, y_t) = \mathbb{P}^*[r_{t+1} = 0|r_t, y_t] &= \frac{\beta'_0(r_t, y_t)\int \kappa\alpha_0}{\beta'(r_t, y_t)\int \kappa\alpha} \\ &= \frac{p(r_t, y_t)\gamma'_0(r_t, y_t)\int \kappa\alpha_0}{p(r_t, y_t)\gamma'_0(r_t, y_t)\int \kappa\alpha_0 + [1 - p(r_t, y_t)]\gamma'_1(r_t, y_t)\int \kappa\alpha_1}. \end{aligned} \tag{4.12}$$

Similarly, the risk-neutral analogues of survival functions $S_{00}(h, y_t)$ and $S_{11}(h, r_t, y_t)$ are obtained by replacing, in equations (3.1) and (3.5), the functions by their risk-neutral counterparts.

For instance, we have:

$$S_{00}^*(h, y_t) = \mathbb{P}^*[r_{t+h} = \cdots = r_{t+1} = 0 | r_t = 0, y_t] = \beta_0^{*\prime}(y_t)(\Pi_{00}^*)^{h-1}\mathbb{1}_S, \tag{4.13}$$

where

$$\Pi_{00}^* = \int \alpha_0^*(0, y)\beta_0^{*\prime}(0, y)\mathrm{d}y. \tag{4.14}$$

This risk-neutral probability ${S_{00}}^*(h, y_t)$ corresponds to the price of an insurance contract protecting against the event that the short rate stays at the ZLB up to time $t + h$ (see also Section 4.4 on derivative pricing).

4.2 Bond pricing

Let us denote by $B(t, h)$ the price at t of the riskfree zero-coupon bond with time-to-maturity h. This price is computed given the information available at time t, i.e. $(\underline{r_t}, \underline{y_t})$, or equivalently (r_t, y_t) by the Markov property. It is given by:

$$B(t, h) = \mathbb{E}[m_{t+1} \cdots m_{t+h} | r_t, y_t]. \tag{4.15}$$

We have the following formula for $B(t, h)$:

Proposition 6. *For each $h \geq 1$:*

$$B(t, h) = \frac{e^{-r_t}\beta'(r_t, y_t)}{\beta'(r_t, y_t)\int \kappa\alpha} M_1^{h-1} \int \kappa\alpha, \tag{4.16}$$

where the $(2S \times 2S)$ matrix M_1 is given by:

$$M_1 = \int e^{-r} \frac{\kappa(r, y)\alpha(r, y)\beta'(r, y)}{\beta'(r, y)\int \kappa\alpha} \mathrm{d}\mu(r, y)$$

Proof. See Appendix 1.3. □

Thus, for any t and given factor value (r_t, y_t), the term structure of interest rate is easily accessible up through the computation of a time-invariant matrix M_1 via Monte-Carlo integration.

For any horizon h, the price $B(t, h)$ depends on the factors through $\frac{\beta(r_t, y_t)}{\beta'(r_t, y_t)\int \kappa\alpha}$ only. This vector is linked to the risk-neutral vector of probabilities β^* by a deterministic linear transformation: $\frac{\beta(r_t, y_t)}{\beta'(r_t, y_t)\int \kappa\alpha} = \mathrm{Diag}(\int \kappa\alpha)^{-1}\beta^*(r_t, y_t)$.

Thus the dynamic model leads to a closed form formula for the term structure of the zero-coupon bond prices. The existence of such a formula is important for estimation purpose as discussed in Section 5.3. It avoids the more complicated approximated computation developed in other modelings, such as the eigenfunction expansions [Gorovoi and Linetsky (2004)], finite-difference solutions of partial differential equations [Kim and Singleton (2012)], or intensive simulations [Christensen and Rudebusch (2014)] in the SRM[6].

4.3 Long-term rate

Let us study the long-term asymptotics of the interest rate $r(t,h)$, when h goes to infinity. Since all the entries of matrix M_1 are strictly positive, by Perron-Frobenius theorem, the spectral radius ρ of M_1, that is the largest absolute value of its eigenvalues, corresponds to a simple eigenvalue and all the other eigenvalues have a modulus strictly smaller than ρ. Thus, when h goes to infinity, we have:

$$B(t,h) \sim \rho^{h-1} v_1 u_1', \tag{4.17}$$

where u_1 (resp. v_1) is the normalized right (resp. left) eigenvector associated with eigenvalue ρ. Thus, the long-term zero-coupon yield:

$$r(t,\infty) = -\lim_{h\infty} \frac{1}{h} \log B(t,h),$$

exists and is equal to $-\log \rho$. This limiting rate is independent of the current state variables (r_t, y_t), in particular of the current ZLB or non ZLB state. This result is consistent with the academic literature, which shows that, under the absence of arbitrage, the long-term interest rate is either constant, or non-decreasing [see e.g. El Karoui et al. (1997); Dybvig et al. (1998)].

Let us now check that ρ is always strictly smaller than one in our model, or, in other words, that the long-term interest rate is strictly positive.

Lemma 1. The spectral radius ρ of matrix M_1 is strictly smaller than 1.

Proof. See Appendix 1.4. □

[6] The approximation proposed in Krippner (2013) based on the forward rates is not internally consistent, that is compatible with an arbitrage free model [see also the linearization performed in Wu and Xia (2016) or in Deutsche Bundesbank (2017)].

4.4 European and Asian interest rate derivative pricing

The dynamic EMC model is convenient for pricing European derivatives written on the short term interest rate at any horizon h. We have to compute quantities of the type:

$$\begin{aligned} C(t,h,p) &= \mathbb{E}[m_{t+1}\cdots m_{t+h}g(r_{t+h})|r_t,y_t] \\ &= \frac{e^{-r_t}\beta'(r_t,y_t)}{\beta'(r_t,y_t)\int\kappa\alpha}M_1^{h-1}\int p(r)\kappa(r,y)\alpha(r,y)\mathrm{d}\mu(r,y), \end{aligned} \tag{4.18}$$

where $g(r_{t+h})$ is the payoff at time $t+h$. The expression of the zero-coupon price given in Proposition 6 is a special case of the pricing formula above, in which g is equal to 1.

Let us now consider a path-dependent, Asian-type derivative with pay-off:

$$g(r_{t+1})g(r_{t+2})\cdots g(r_{t+h}). \tag{4.19}$$

The following proposition shows that the pricing of such path-dependent derivatives is as simple as that of European derivatives.

Proposition 7. *The price of the derivative paying* (4.19) *at time* $t+h$ *is:*

$$\mathbb{E}[m_{t+1}\cdots m_{t+h}g(r_{t+1})g(r_{t+2})\cdots g(r_{t+h})|r_t,y_t] = \frac{e^{-r_t}\beta'(r_t,y_t)}{\beta'(r_t,y_t)\int\kappa\alpha}M_2^{h-1}(g)\int g\kappa\alpha, \tag{4.20}$$

where

$$M_2(g) = \int e^{-r}\frac{\kappa(r,y)g(r)\alpha(r,y)\beta'(r,y)}{\beta'(r,y)\int\kappa\alpha}\mathrm{d}\mu(r,y).$$

The proof is similar to that of Proposition 6 and is therefore omitted.

In particular, this formula can be used to price any derivative written on the remaining time to be spent at the ZLB. Indeed, let us consider $g(r) = \mathbb{1}_{r=0}$, and take a time t such that $r_t = 0$, then we get:

$$\mathbb{E}[m_{t+1}\cdots m_{t+h}\mathbb{1}_{r_{t+1}=0}\mathbb{1}_{r_{t+2}=0}\cdots\mathbb{1}_{r_{t+h}=0}|r_t=0,y_t] = \mathbb{P}^*[r_{t+h}=r_{t+h-1}=\cdots=r_{t+1}=0|r_t=0,y_t],$$

which is the risk-neutral survival probability $S^*_{00}(h,y_t)$ given in (4.13). Similarly, if we take $g(r) = \mathbb{1}_{r>0}$, and a time t such that $r_t > 0$, then we get the expression of $S^*_{11}(h,r_t,y_t)$.

5 A constrained specification for implementation

Let us now explain how the EMC modelling can be implemented in practice. We first discuss the choice of latent factors y_t. More precisely, it is important for the interpretation, estimation and stress purposes to disentangle the factors driving the dynamics under the ZLB and outside the ZLB regime, respectively. This leads to constrained dynamic specification that also facilitates statistical inference.

5.1 Separating factors specific to the ZLB and non-ZLB states

Compared with the existing term structure models with ZLB, such as the shadow rate model, or the affine model based on autoregressive gamma-zero dynamics, the EMC model is much more flexible. Let us for instance consider a decomposition of $y_t = (y'_{0t}, y'_{1t})'$ into two subvectors, introduced to manage the dynamics under (resp. outside) the ZLB state. In other words, we allow for level, slope,..., factors for each regime, on the contrary to the standard shadow rate model, in which these factors are defined for the shadow rate and do not depend on the regime [see e.g. Christensen (2015); Christensen and Rudebusch (2015); Carriero et al. (2015)]. More precisely, we specify the conditional probabilities β as:

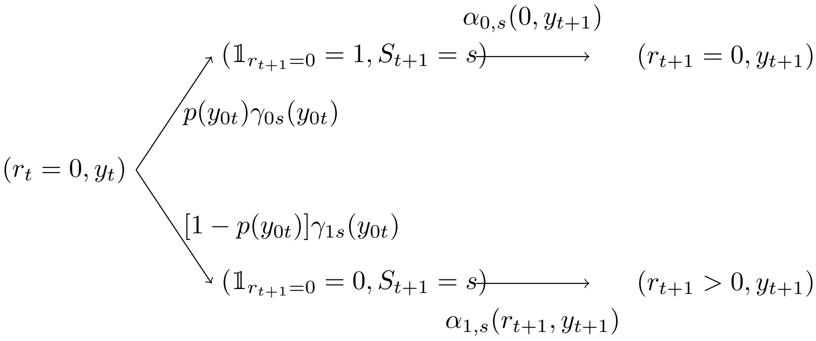
Assumption 2. *If $r_t = 0$, then*

$$\beta(r_t, y_t) = \beta(y_{0t}) = \begin{bmatrix} p(y_{0t})\gamma_0(y_{0t}) \\ [1 - p(y_{0t})]\gamma_1(y_{0t}) \end{bmatrix}, \tag{5.1}$$

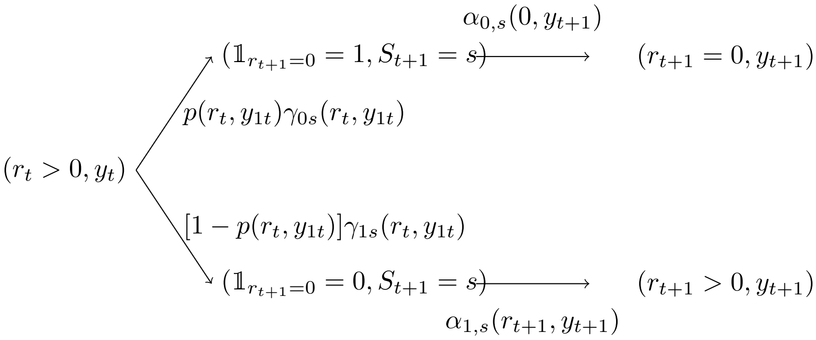
and, if $r_t > 0$, then

$$\beta(r_t, y_t) = \beta(r_t, y_{1t}) = \begin{bmatrix} p(r_t, y_{1t})\gamma_0(r_t, y_{1t}) \\ [1 - p(r_t, y_{1t})]\gamma_1(r_t, y_{1t}) \end{bmatrix}. \tag{5.2}$$

Under this assumption, we can rewrite the causal chain (2.5) according to the value of $\mathbb{1}_{r_t}$. If $r_t = 0$, we get:



whereas if if $r_t > 0$, the causal scheme becomes:



A first consequence of this assumption concerns the (direct or indirect observability) of the factors, when zero-coupon prices (say) are observable. First, the short rate r_t is always directly observable on the market. Then by looking at the pricing formula in Section 4.2, either only y_{0t} is recoverable from the zero coupon prices (if $r_t = 0$), or only y_{1t} is recoverable (if $r_t > 0$). Thus the recoverable factors, F_t are given by:

$$F_t = \begin{cases} (r_t, y'_{0t})', & \text{if } r_t = 0, \\ (r_t, y'_{1t})', & \text{if } r_t > 0. \end{cases}$$

Would this impossibility to recover the other component y_{1t} (resp. y_{0t}) in (resp. out of) the ZLB regime be detrimental for risk analysis? The answer is no. Indeed we have the following corollary, which is a direct consequence of Assumption 2.

Corollary 4. *Under Assumption 2, the recoverable factor* (F_t) *is itself a Markov process with* *respect to its own history. Moreover the conditional density of factor* (r_t, y_t) *is given by:*

$$f(r_{t+1}, y_{t+1}|r_t, y_t) = \begin{cases} f(r_{t+1}, y_{t+1}|0, y_{0t}), & \textit{if } r_t = 0, \\ f(r_{t+1}, y_{t+1}|r_t, y_{1t}), & \textit{if } r_t > 0. \end{cases} \tag{5.3}$$

Proof. By applying the causal scheme (2.5), we get under the ZLB regime:

$$f(0, y_{t+1}|0, y_t) = p(y_{0t}) \sum_{s=1}^{S} \gamma_{0,s}(y_{0,t}) \alpha_{0,s}(0, y_{t+1}), \tag{5.4}$$

$$f(r_{t+1}, y_{t+1}|0, y_t) = [1 - p(y_{0t})] \sum_{s=1}^{S} \gamma_{1,s}(y_{0,t}) \alpha_{0,s}(r_{t+1}, y_{t+1}). \tag{5.5}$$

Both densities depend on the past through y_{0t} only. A similar result is derived under the non ZLB regime. □

In other words, the unrecoverable components of the factor does not Granger cause the future value of the recoverable factor.

Let us now consider the risk-neutral conditional densities, for instance in the ZLB regime. Under Assumption 2, the conditional risk-neutral distributions of the state variable given in (4.7) becomes:

$$\beta^*_{0,s}(0, y_t) = \frac{p(y_{0t})\gamma_{0s}(y_{0t}) \int \kappa\alpha_{0,s}}{p(y_{0t}) \sum_{j=1}^{S} \gamma_{0j}(y_{0t}) \int \kappa\alpha_{0,j} + [1 - p(y_{0t})] \sum_{j=1}^{S} \gamma_{1j}(y_{0t}) \int \kappa\alpha_{1,j}}, \tag{5.6}$$

$$\beta^*_{1,s}(0, y_t) = \frac{[1 - p(y_{0t})]\gamma_{1s}(y_{0t}) \int \kappa\alpha_{1,s}}{p(y_{0t}) \sum_{j=1}^{S} \gamma_{0j}(y_{0t}) \int \kappa\alpha_{0,j} + [1 - p(y_{0t})] \sum_{j=1}^{S} \gamma_{1j}(y_{0t}) \int \kappa\alpha_{1,j}}. \tag{5.7}$$

We deduce the following corollary:

Corollary 5. *Under Assumption 2, we get:*

If $r_t = 0$*:*

$$\beta^*(r_t, y_t) = \beta^*(0, y_{0t}) = \begin{bmatrix} p^*(y_{0t})\gamma^*_0(y_{0,t}) \\ [1 - p^*(y_{0t})]\gamma^*_1(y_{0,t}) \end{bmatrix},$$

and if $r_t > 0$*, then:*

$$\beta^*(r_t, y_t) = \beta^*(r_t, y_{1t}) = \begin{bmatrix} p^*(r_t, y_{1t})\gamma^*_0(r_t, y_{1t}) \\ [1 - p^*(r_t, y_{1t})]\gamma^*_1(r_t, y_{1t}) \end{bmatrix},$$

where

$$p^*(y_{0t}) = \frac{p(y_{0t}) \sum_{j=1}^{S} \gamma_{0j}(y_{0t}) \int \kappa\alpha_{0,j}}{p(y_{0t}) \sum_{j=1}^{S} \gamma_{0j}(y_{0t}) \int \kappa\alpha_{0,j} + [1 - p(y_{0t})] \sum_{j=1}^{S} \gamma_{1j}(y_{0t}) \int \kappa\alpha_{1,j}},$$
$$\gamma^*_{0,s}(y_{0,t}) = \frac{\gamma_{0s}(y_{0t}) \int \kappa\alpha_{0,s}}{\sum_{j=1}^{S} \gamma_{0j}(y_{0t}) \int \kappa\alpha_{0,j}}, \qquad \gamma^*_{1,s}(y_{0,t}) = \frac{\gamma_{1s}(y_{0t}) \int \kappa\alpha_{1,s}}{\sum_{j=1}^{S} \gamma_{1j}(y_{0t}) \int \kappa\alpha_{1,j}},$$

and similarly for $p^*(r_t, y_{1t})$, $\gamma^*_{0,s}(r_t, y_{1,t})$ *and* $\gamma^*_{1,s}(r_t, y_{1,t})$.

As a consequence, we get a similar noncausality property for the risk-neutral dynamics, that is:

$$f^*(r_{t+1}, y_{t+1}|r_t, y_t) = \begin{cases} f^*(r_{t+1}, y_{t+1}|0, y_{0t}), & \text{if } r_t = 0, \\ f^*(r_{t+1}, y_{t+1}|r_t, y_{1t}), & \text{if } r_t > 0. \end{cases} \tag{5.8}$$

To summarize we have the following proposition:

Proposition 8. *Under Assumption 2, both the historical density forecasts [eq.* (2.12)*] and the risk-neutral derivative prices [eq.* (4.16)*] depend only on* y_{0t} *in the ZLB state, or only on* (r_t, y_{1t}) *in the non-ZLB state.*

Thus, the unrecoverable factor components provides no extra useful information for forecasting or pricing. For instance, if we are currently at the ZLB, r_t is equal to zero, y_{1t} is recovered, y_{1t} is unrecoverable; at this date the economist is interested in the scenario that the short rate leaves the ZLB in the next period, in which case, the bond prices quoted at the next period will depend on $r_{t+1} > 0$ and $y_{1,t+1}$. Nevertheless to predict these future bond prices, it is not necessary to "filter out" the unobservable factor y_{1t}, as the recoverable process y_{0t} contains all the sufficient information to forecast $y_{1,t+1}$ and these future bond prices.

The unrecoverable factor component in our model can be regarded as a kind of unobserved heterogeneity. However this unobserved heterogeneity cannot be omitted without loss of generality, that is, the model is not observationally equivalent to a model without unobserved heterogeneity. Indeed, the sdf specification in (4.3) depends on all the components of y_t. Thus matrix M_1 in the bond pricing formula, as well as the risk-neutral dynamics of the recoverable part depends also on the dynamics of the unrecoverable factor components, although the conditional distribution of the unrecoverable factor depends on the recoverable components only.

Remark 2. The noncausality property induced by Assumption 2 does not imply a conditional independence between the recoverable and unrecoverable factors. Let us for instance consider the transition (5.4). This conditional density $f(0, y_{t+1}|0, y_t)$ cannot be written in general as the product of a function of $y_{0,t+1}$ and a function of $y_{1,t+1}$. This dependence is due to two effects: first $y_{0,t+1}$ and $y_{1,t+1}$ can be dependent given $S_{t+1} = s$, for any s. Second, even if they were independent given S_{t+1}, there is the mixture effect of state S_{t+1}.

Due to this dependence, even if only y_{0t} is recoverable under the ZLB regime, the observation of y_{0t} will provide useful information on the unrecoverable y_{1t}.

5.2 A Monte Carlo illustration

In the following, we propose a parametric specification of the model under Assumption 2 and simulate trajectories of the term structure of interest rates. First, we assume that:

Assumption 3. *We have, for any values of r_t, y_t:*

$$\gamma_0(r_t, y_t) = \gamma_1(r_t, y_t) := \gamma(r_t, y_t). \tag{5.9}$$

Let us remind that by definition, for each $s = 1, ..., S$, we have:

$$\gamma_{0,s}(r_t, y_t) = \mathbb{P}[S_{t+1} = s | r_{t+1} = 0, r_t, y_t], \qquad \gamma_{1,s}(r_t, y_t) = \mathbb{P}[S_{t+1} = s | r_{t+1} > 0, r_t, y_t].$$

Thus Assumption 3 is equivalent to the independence between the unobservable future regime S_{t+1} and the other observable future regime variable $\mathbb{1}_{r_{t+1}>0}$, given r_t, y_t.

This conditional independence should not be understood as the independence between S_{t+1} and the current regime variable $\mathbb{1}_{r_t>0}$. Instead, the form of function $\gamma(r_t, y_t)$ in equation (5.9) continues to depend on the current state $\mathbb{1}_{r_t>0}$, in the sense that it does not depend on y_{1t} (resp. y_{0t}) if the short rate is at (resp. outside) the ZLB.

Assumption 4. *1. (Dimension of y_t) Factors y_{0t}, y_{1t} are both of dimension S:*

$$y_{0t} = (y_{0t,0}, y_{0t,1}, ..., y_{0t,S-1})', \qquad y_{1t} = (y_{1t,0}, y_{1t,1}, ..., y_{1t,S-1})'.$$

2. (Specification of κ) Function κ in the sdf has the exponential affine form under each regime:

$$\kappa(r_t, y_t) = \exp(d_2' y_{0t}), \qquad \textit{if} \qquad r_t = 0, \tag{5.10}$$

and

$$\kappa(r_t, y_t) = \exp(d_1 r_t + d_2' y_{1t}), \qquad \text{if} \qquad r_t > 0, \tag{5.11}$$

3. *(Specification of p):*

$$p(0, r_t, y_t) = \begin{cases} \frac{1}{1+e^{\lambda_1+\lambda_2 y_{0t,0}}}, & \text{if} \quad r_t = 0, \\ \frac{1}{1+e^{\lambda_1+\lambda_2 y_{1t,0}+\lambda_3 r_t}} & \text{if} \quad r_t > 0. \end{cases}, \tag{5.12}$$

where $y_{0t,0}$ *(resp.* $y_{1t,0}$*) is the first component of* y_{0t} *(resp.* y_{1t}*).*

4. *(Specification of* γ*) When* $r_t = 0$*,*

$$\gamma(0, y_{0t}) = \Big(\frac{1}{1+\sum_{j=1}^{S-1} e^{c_{1,j} y_{0t,j}+c_{2,j}}}, ..., \frac{e^{c_{1,S-1} y_{0t,S-1}+c_{2,S-1}}}{1+\sum_{j=1}^{S-1} e^{c_{1,j} y_{0t,j}+c_{2,j}}}\Big), \tag{5.13}$$

and when $r_t > 0$*,*

$$\gamma(r_t, y_{1t}) = \Big(\frac{1}{1+\sum_{j=1}^{S-1} e^{c_{3,j} r_t+c_{4,j} y_{1t,j}+c_{5,j}}}, ..., \frac{e^{c_{3,S-1} r_t+c_{4,S-1} y_{1t,S-1}+c_{5,S-1}}}{1+\sum_{j=1}^{S-1} e^{c_{3,j} r_t+c_{4,j} y_{1t,j}+c_{5,j}}}\Big), \tag{5.14}$$

Thus, we assume that $p(r_t, y_t)$ and $\gamma(r_t, y_t)$ depend on different components of vector y_t. This simplifying assumption is motivated by the duration analysis of Section 3, which shows that $p(r_t, y_t)$ and $\gamma(r_t, y_t)$ characterize the spot and forward instantaneous survival probability of the ZLB regime indicator $\mathbb{1}_{r_t>0}$, respectively. The first factor component $y_{0t,0}$ (resp. $y_{1t,0}$) drives the spot instantaneous probability and the other components drive their forward counterparts. We can also remark that, we can apply to each component of y_{0t}, y_{1t} any affine transformations, and then change the multiplicative and additive parameters accordingly, without changing the model. Thus, without loss of generality, we can assume that $c_{1,j} = c_{4,j} = 1, c_{2,j} = c_{5,j} = 0, j = 0, ..., S-1$.

Finally, let us consider the specification of conditional distributions of components of y_t given the regimes, characterized by conditional densities $\alpha_{j,s}$, $j = 0, 1$, $s = 1, ..., S$.

We make the following assumption:

Assumption 5. *Given the state variable* $S_t = s$*,*

1. y_{0t}*,* y_{1t} *are (conditionally) independent from* r_{t+1}*;*

2. *all the components of* y_{0t} *and of* y_{1t} *are independent, normally distributed with unitary*

variance σ_y and mean $(\mu_0, \mu_1)_{j,s}$ satisfying:

$$\mu_{0,j,s} = \mu_0 + \delta_0(j+s), \qquad \forall s = 1, ...S, j = 1, ..., S-1,$$

$$\mu_{1,j,s} = \mu_1 + \delta_1(j+s), \qquad \forall s = 1, ...S, j = 1, ..., S-1.$$

Thus the larger the regime variable $S_t = s$, the larger the expected value of each components of y.

3. *if $r_{t+1} > 0$, then $\log r_{t+1}$ follows normal distribution with variance σ_r and mean:*

$$\mu_s = \mu_{r,0} + \delta_r s, \qquad \forall s = 1, ...S$$

Thus the set of parameters can be decomposed as:

- 3 parameters characterizing the spot survival probabilities p: $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.
- 10 parameters characterizing the conditional distribution of y_t and r_t in the different regimes: $\mu_{0,0}, \mu_{1,0}, \mu_0, \mu_1, \delta_0, \delta_1, \mu_r, \delta_r$, σ_y, σ_r.
- $S+1$ parameters characterizing the sdf κ: $d_1 \in \mathbb{R}, d_2 \in \mathbb{R}^S$.
- $S-1$ parameters characterizing the sequence $(c_{3,j})_{j=1}^{S-1}$.

Thus the total number of parameters is $2S+13$. For the illustration below, we use a model with $S=2$, i.e. $2S=4$ underlying states, and the dimensions of y_{0t}, y_{1t} both equal to 2. Then the parametric model contains 17 parameters. Their values have been fixed according to Table 1 below.

Parameter	λ_1	λ_2	λ_3	μ_0	μ_1	δ_0	δ_1	μ_r	$\mu_{0,0}$
Value	-2	0.05							
Parameter	$\mu_{1,0}$	δ_r	d_1	$d_{2,1}$	$d_{2,2}$	σ_r	σ_y	$c_{3,1}$	
Value									

Table 1: Values of parameters.

In the following Figure 1 we plot a simulated path of the short rate process, accompanied by its autocorrelation function (ACF).

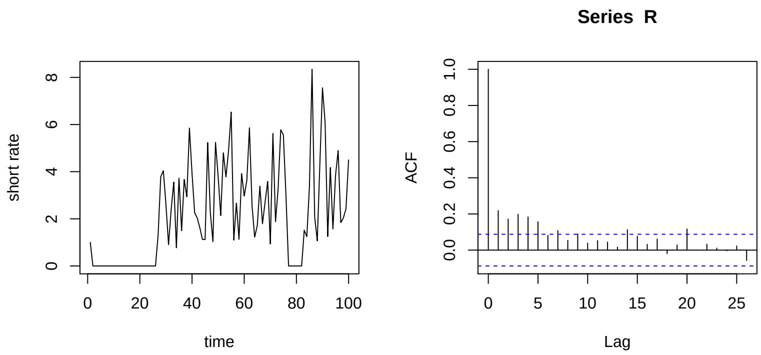


Figure 1: Simulated path of the process of short rate as well as the corresponding ACF.

We can see that there are two prolonged periods where the short rate is at the ZLB state. The next figures provide examples of the term structure of yield rate for two different dates when the short rate is at the ZLB. In the simulation we have used a model with $2S = 6$ different states, and we can see that the model is quite flexible to allow for three different yield curves.

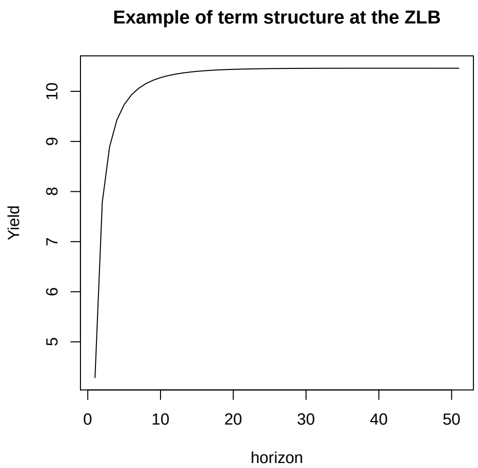


Figure 2: Example of an increasing yield curve

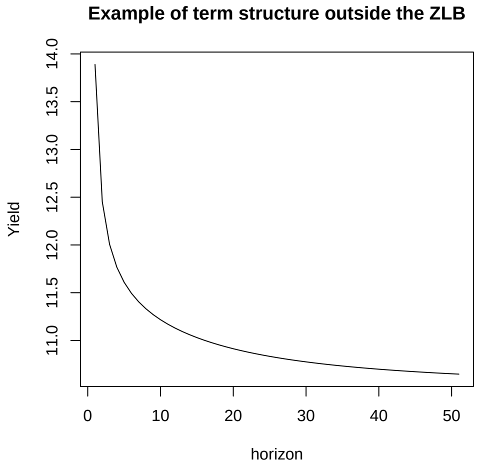


Figure 3: Example of an inversed yield curve

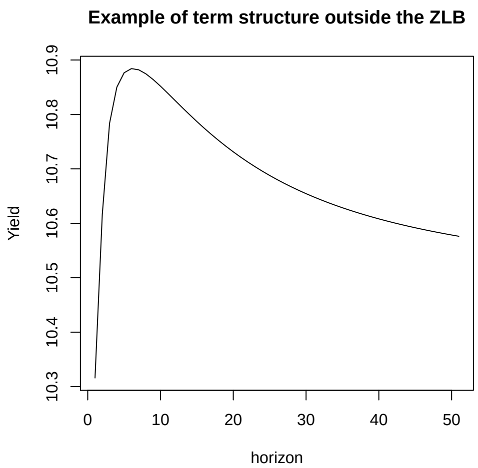


Figure 4: Example of a humped yield curve

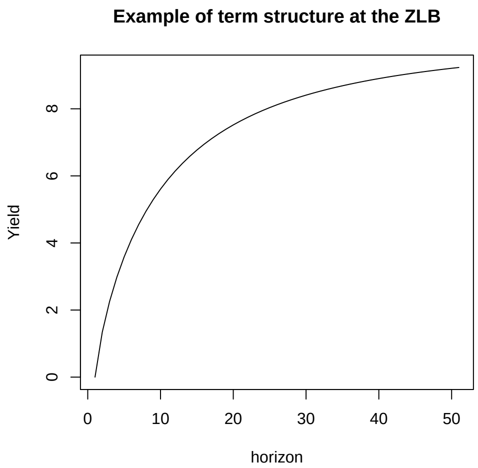


Figure 5: Example of yield curve when the short rate is at the ZLB.

5.3 Estimation procedure

Let us consider the estimation of the EMC model under Assumptions 1-2. We denote by θ the vector of parameters that includes the parameters characterizing the historical dynamics and the sdf (see Section 5.3 for the list of parameters). We assume that, at each date $t = 1, ..., T$, we observe the short term rate r_t and $S = g$ other rates corresponding to different maturities, where $g = \dim(y_{0t}) = \dim(y_{1t})$. These other rates are gathered in a vector R_t. From the pricing subsection 4.2, we know that these rates are deterministic functions of the underlying factors r_t and y_t. More precisely,

$$\text{for a date } t \text{ in the ZLB regime, } r_t = 0, \text{ we have: } R_t = \tilde{R}_t(y_{0t}, \theta), \tag{5.15}$$

$$\text{for a date } t \text{ out of the ZLB regime, } r_t > 0, \text{ we have: } R_t = \tilde{R}_t(r_t, y_{1t}, \theta). \tag{5.16}$$

We assume that the additional observed rates are fully informative in the sense that the mappings:

$$y_{0t} \mapsto \tilde{R}_t(y_{0t}, \theta), \qquad \text{and } y_{1t} \mapsto \tilde{R}_t(r_t, y_{1t}, \theta),$$

are one-to-one.

Thus we do not assume that the underlying factors are directly observable, for instance chosen a priori to correspond to the Asset Purchase Program (APP) holdings[7], or to the Long-Term

[7] Possibly distinguishing the asset-backed securities purchase program and the covered bond purchase program.

Refinancing Operation (LTRO) of a Central Bank[8]. Instead we reconstitute indirectly factor values from the observed term structure of interest rates only. These dynamic filtered factor values can then be linked with the policy instruments such as APP or LTRO of a Central Bank.

This estimation approach below is an exact maximum likelihood method appropriate when the observations are derivative prices [see Pastorello et al. (2000) for the first implementation of this technique for stochastic volatility models]. It avoids the use of the extended Kalman filter usually employed in the SRM [Christensen and Rudebusch (2015) or in the ARG-zero model of Monfort et al. (2017), which usually induces efficiency loss.

We can easily check that under the multinomial logit specification, the pricing functions in (5.15) are one-to-one. Indeed, the bond price is a linear function of $\frac{\beta(r_t,y_t)}{\beta'(r_t,y_t)\int\kappa\alpha}$, where $\beta(r_t, y_t) = [p(x_t)\gamma_0(x_t)', (1-p(x_t))\gamma_1'(x_t)]'$, or $\beta(r_t, y_t) = [p(x_t)\gamma_0(x_t)', (1-p(x_t))\gamma_0'(x_t)]'$ under Assumption 3. Thus it is up to a normalization term depending on γ_0, a linear function of $\gamma_0(x_t)$. Thus if we observe the price of exactly S different zero-coupon bonds, where S is equal to the dimension of γ_0, then by solving a linear equation, we can recover in a unique way γ_0, as well as the constant. Then the values of y_{0t} (or y_{1t}, depending on the regime can be recovered uniquely. Finally the normalization constant is uniquely recovered, allowing for the identification of η_t. We can remark that the whole process only involves linear algebra and elementary functions.

To derive the expression of the likelihood, we proceed in three steps.

Step 1: if all factors were observable, the likelihood function would be:

$$\ell^{**}(r, y, \theta) = \prod_{t=2}^{T} l(r_t, y_t | r_t, y_{t-1}, \theta).$$

This latent likelihood can be decomposed according to the different regime transitions. Let us denote by $T_{i,j}$, $i, j = 0, 1$, the subsets of dates t, where r_{t+1} is in regime i and r_t is in regime j (where $i = 0$ means $r_t = 0$ and $i = 1$ means $r_t > 0$). We get:

$$\ell^{**}(r, y, \theta) = \prod_{t\in T_{00}} l(r_t, y_t | r_t, y_{t-1}, \theta) \prod_{t\in T_{01}} l(r_t, y_t | r_t, y_{t-1}, \theta) \prod_{t\in T_{10}} l(r_t, y_t | r_t, y_{t-1}, \theta) \prod_{t\in T_{11}} l(r_t, y_t | r_t, y_{t-1}, \theta). \tag{5.17}$$

Step 2: But only y_{0t} can be recovered in regime 0 and y_{1t} can be recovered in regime 1. Under the assumption that function β only depends either on y_{0t} or r_t, y_{1t} according to the regime, we

[8] Or to the level and slope factors estimated from a misspecified Gaussian affine model [Carriero et al. (2015)]

have:

$$\ell^*(r,y,\theta) = \prod_{t\in T_{00}} l(y_{0t}|\{r_{t-1}=0\}, y_{0,t-1},\theta) \prod_{t\in T_{01}} l(r_t, y_{1t}|\{r_{t-1}=0\}, y_{0,t-1},\theta)$$
$$\times \prod_{t\in T_{10}} l(y_{0t}|\{r_{t-1}>0\}, r_{t-1}, y_{1,t-1},\theta) \prod_{t\in T_{11}} l(r_t, y_{1t}|\{r_{t-1}>0\}, r_{t-1}, y_{1,t-1},\theta). \tag{5.18}$$

Step 3: Finally we have to take into account the fact that the effective factors y_{0t}, $t \in T_{01}\cup T_{00}$ and y_{1t}, $t \in T_{11}\cup T_{10}$, are not directly observable. They are known through the pricing functions $\tilde{R}_t$ that involve unknown parameters. Therefore we introduce the appropriate Jacobian term, and:

$$\ell(r,y,\theta) = \ell^*(r,y,\theta)\Pi_{t\in T_{11}\cup T_{10}}\Big(\det \frac{\partial(r_t,R_t)}{\partial(r_t,y_{1t})}\Big)^{-1}\Pi_{t\in T_{01}\cup T_{00}}\Big(\det \frac{\partial(r_t,R_t)}{\partial(r_t,y_{0t})}\Big)^{-1}. \tag{5.19}$$

The maximum likelihood estimator of θ is obtained by maximizing the log-likelihood corresponding to (5.19). Once θ is estimated, the "observable" factor values are deduced by inverting the relations (5.1)-(5.2), after replacing the unknown parameter values by its maximum likelihood estimates.

5.4 Estimation on simulated data

6 Conclusion

Appendix 1 Proofs

Appendix 1.1 Proof of Proposition 3.

Let us first write the joint conditional density of $r_{t+h}, y_{t+h}, r_{t+h-1}, y_{t+h-1}, \cdots, r_{t+1}, y_{t+1}$ given (r_t, y_t). By the Markov property we have:

$$\ell(r_{t+h}, y_{t+h}, r_{t+h-1}, y_{t+h-1}, \cdots, r_{t+1}, y_{t+1}|r_t=0, y_t)$$
$$= \beta'(r_t,y_t)\alpha(r_{t+1},y_{t+1})\beta'(r_{t+1},y_{t+1})\alpha(r_{t+2},y_{t+2})\cdots\beta'(r_{t+h-1},y_{t+h-1})\alpha(r_{t+h},y_{t+h}).$$

When $r_t = r_{t+1} = r_{t+2} = \cdots = r_{t+h} = 0$, this joint density becomes:

$$\ell(0, y_{t+h}, 0, y_{t+h-1}, \cdots, 0, y_{t+1}|r_t, y_t) = \beta_0'(0, y_t)\alpha_0(0, y_{t+1})\beta_0'(0, y_{t+1})\alpha_0(0, y_{t+2}) \cdots \beta_0'(0, y_{t+h-1})\alpha_0(0, y_{t+h}).$$

By integrating out all intermediate variables $y_{t+1}, ..., y_{t+h}$ with respect to $\mathrm{d}y_{t+1}, ..., \mathrm{d}y_{t+h}$, we get the conditional probability:

$$S_{00}(h, y_t) = \ell(0, 0, \cdots, 0|r_t = 0, y_t) = \beta_0'(0, y_t)\Pi_{00}^{h-1}\mathbb{1}_S.$$

Appendix 1.2 Proof of Proposition 5

We have:

$$\begin{aligned}
&\mathbb{E}[e^{-uD(t,h)}|r_t, y_t] \\
&= \mathbb{E}[\exp(-u\mathbb{1}_{r_{t+1}=0} - \cdots - u\mathbb{1}_{r_{t+h}=0})|r_t, y_t] && \text{(eq. a.1)} \\
&= \int \exp(-u\mathbb{1}_{r_{t+1}=0} - \cdots - u\mathbb{1}_{r_{t+h}=0})\ell(r_{t+h}, y_{t+h}, r_{t+h-1}, y_{t+h-1}, \cdots, r_{t+1}, y_{t+1}|r_t, y_t) \\
&\qquad \mathrm{d}\mu(r_{t+1}, y_{t+1})\mathrm{d}\mu(r_{t+2}, y_{t+2}) \cdots \mathrm{d}\mu(r_{t+h}, y_{t+h}) && \text{(eq. a.2)} \\
&= \beta'(r_t, y_t) \int \exp(-u\mathbb{1}_{r_{t+1}=0})\alpha(r_{t+1}, y_{t+1})\beta'(r_{t+1}, y_{t+1})\alpha(r_{t+2}, y_{t+2}) \exp(-u\mathbb{1}_{r_{t+2}=0})\beta'(r_{t+2}, y_{t+2}) \\
&\qquad \cdots \times \mathbb{1}_{r_{t+h}=0}\mathrm{d}\mu(r_{t+1}, y_{t+1})\mathrm{d}\mu(r_{t+2}, y_{t+2}) \cdots \mathrm{d}\mu(r_{t+h}, y_{t+h}) && \text{(eq. a.3)} \\
&= \beta'(r_t, y_t)\Big[\int \exp(-u\mathbb{1}_{r=0})\alpha(r, y)\beta'(r, y)\mathrm{d}\mu(r, y)\Big]^{h-1} \int \exp(-u\mathbb{1}_{r=0})\alpha(r, y)\mathrm{d}\mu(r, y) && \text{(eq. a.4)} \\
&= \beta'(r_t, y_t)M(u)^{h-1} \begin{bmatrix} \exp(-u)\mathbb{1}_S \\ \mathbb{1}_S \end{bmatrix}, && \text{(eq. a.5)}
\end{aligned}$$

where from equation (eq. a.1) to (eq. a.2) we have integrated with respect to the conditional joint distribution of $(r_{t+1}, y_{t+1}, ..., r_{t+h}, y_{t+h})$ given (r_t, y_t).

Appendix 1.3 Proof of Proposition 6

The proof follows the same principle as the proof of Proposition 5. We have:

$$
\begin{aligned}
B(t,h) &= \mathbb{E}[m_{t+1}\cdots m_{t+h}|r_t, y_t] \\
&= \mathbb{E}\Big[\frac{\kappa(r_{t+1},y_{t+1})}{\beta'(r_t,y_t)\int\kappa\alpha}e^{-r_t}\times\cdots\times\frac{\kappa(r_{t+h},y_{t+h})}{\beta'(r_{t+h-1},y_{t+h-1})\int\kappa\alpha}e^{-r_{t+h-1}}|r_t,y_t\Big] \\
&= \frac{\exp(-r_t)\beta'(r_t,y_t)}{\beta'(r_t,y_t)\int\kappa\alpha}M_1^{h-1}\int\kappa\alpha.
\end{aligned}
$$

Appendix 1.4 Proof of Lemma 1

We have:

$$
\begin{aligned}
M_1\int\kappa\alpha &= \int\exp(-r)\frac{\kappa(r,y)\alpha(r,y)\beta'(r,y)\int\kappa\alpha}{\beta'(r,y)\int\kappa\alpha}\mathrm{d}\mu(r,y) \\
&= \int\exp(-r)\kappa(r,y)\alpha(r,y)\mathrm{d}\mu(r,y) \leq \int\kappa(r,y)\alpha(r,y)\mathrm{d}\mu(r,y) = \int\kappa\alpha,
\end{aligned}
$$

where the inequality holds componentwise. More precisely, for the S first components, we have equality between the two terms, whereas for the other S components, the inequality is strict.

Since all the entries of M_1, $\int\kappa\alpha$ and $M_1\int\kappa\alpha$ are positive, we can show that:

$$
M_1^2\int\kappa\alpha < M_1^1\int\kappa\alpha,
$$

where the inequalities are strict for each component. Then we can introduce the constant

$$
c = \max_{i=1}^{2S}\frac{M_1\int\kappa\alpha_i}{(M_1^2\int\kappa\alpha)_i} \in (0,1),
$$

where $(M_1\int\kappa\alpha)_i$ is the $i-$th component of vector $M_1\int\kappa\alpha$. This constant satisfies, by induction:

$$
M_1^{h+1}\int\kappa\alpha \leq c^h M_1\int\kappa\alpha, \qquad \forall h\geq 1, \tag{eq. a.6}
$$

where the inequalities are again componentwise. On the other hand, asymptotically, each component of $M_1^h\int\kappa$ behaves as ρ^h times a constant, where ρ is the spectral radius[9] of M_1. By equation (eq. a.6), we deduce that $\rho\leq c<1$.

[9]Indeed, matrix M_1 has only positive entries. Thus by Perron-Frobenius theorem, its spectral radius is a simple eigenvalue.

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